THE MATHEMATICS TEACHER

Volume XLVI NOVEMBER · 1953 Number 7

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OFFICIAL JOURNAL OF THE
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth Street, N.W., Washington 6, D.C.

Printed at Menasha, Wisconsin, U.S.A.

Entered as second-class matter at the post office at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 412 P. L. & R., authorized March 1, 1930. Printed in U.S.A.

THE MATHEMATICS TEACHER

Official Journal of the National Council of Teachers of Mathematics

Devoted to the interests of mathematics teachers in Elementary and Secondary Schools, Junior Colleges and Teacher Education

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THE MATHEMATICS TEACHER

Volume XLVI — Number 7

November — 1953

Topology for Secondary Schools*

By BRUCE E. MESERVE
University of Illinois, Urbana, Illinois

I SPEAK very humbly of topology for I am not a topologist. Indeed, I speak because of the default of the topologists. When a topologist is asked a mathematical question, he will frequently demonstrate his use of visual images. For example, about a year ago I asked a question regarding properties of surfaces and was told to visualize the Hawaiian Islands as mountains on dry land and to consider the effect of the sea level gradually rising until the islands were completely submerged. Many topologists use such visual images but they have not interpreted their subject to us in this way. I shall use a visual image approach in presenting a few very elementary but fundamental topological concepts.

The word "topology" undoubtedly has very vague connotations for most teachers of mathematics. I shall endeavor to bring the word to life for you with an intuitive understanding of topology as a geometry with some of the properties of ordinary geometry—Euclidean geometry. I shall endeavor to convince you that topology has significance and meaning for mathematics teachers and for pupils in secondary schools.

There appears to be a trend towards the teaching of the underlying principles of any subject. Sometimes these principles are considered in the sense of the basic postulates or assumptions. They form the core of the subject, the underlying reasons for all its properties and theorems. For example, the function concept is one of the fundamental concepts of algebra. The area of a square depends upon the length of the sides of the square and is said to be a function of that length. The area of a circle is a function of its radius. The time of day is a function of the relative position of the sun. Your impatience with my introductory remarks is a function of your understanding of them. Functional dependence is an underlying principle of all mathematics and of logic. It must be recognized and emphasized. I am a bit ashamed to report that while I was visiting a high school recently one mathematics teacher questioned the inclusion of the function concept as a topic and stated that she had never taught the function concept. I think that it would be more correct to say that she had never recognized the function concept when she taught it. As teachers of mathematics we must all strive to recognize and emphasize the fundamental principles of mathematics. Only in this way can we impart the power and effectiveness of mathematics to our students.

Continuity is another fundamental concept of mathematics. It is *the* basic property of topology. It is also a basic property of high school algebra and geometry.

* Presented October 18, 1952 at the Twenty-Second Annual Conference of Teachers of Mathematics, Iowa City, Iowa.

In algebra continuity underlies the extension of the set of rational numbers to the set of real numbers.† It enables us to say that any polynomial $p(x)$ that is negative when $x=a$ and positive when $x=b$ must have at least one zero on the interval (a, b) (Fig. 1). Accordingly continuity is

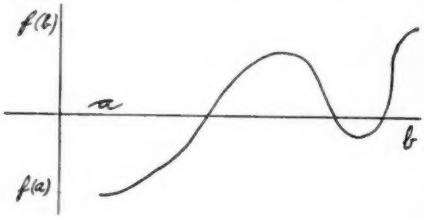

Fig. 1

the principle underlying our intuitive idea that there are no holes in the line, that one cannot cross a line without passing through a point of the line. Euclid tacitly assumed continuity in the sense that there are no holes in a circle. He assumed that any line segment joining the center O of a circle to a point P outside the circle must contain a point of the circle (Fig. 2). Thus continuity is one of the

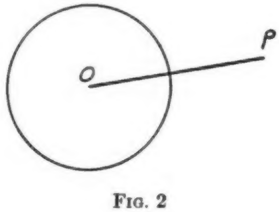

Fig. 2

fundamental concepts of Euclidean geometry.

Topology includes the study of the properties of Euclidean geometry that depend upon continuity. Formally, topology is the study of properties left invariant under continuous transformations. Before discussing topology let us consider a few transformations and clarify what is meant by saying that a property is invariant under a transformation or under a set of transformations.

Ordinary geometry—Euclidean geometry—is the geometry of rigid motions, that is, Euclidean geometry is the study of properties that are invariant under rigid motions. Any rigid motion on a Euclidean plane may be expressed as a transformation of points (x, y) to points (x', y') where

$$x' = x \cos\theta - y \sin\theta + a,$$
$$y' = x \sin\theta + y \cos\theta + b.$$

In other words, any rigid motion on a plane may be expressed in terms of a rotation

$$x' = x \cos\theta - y \sin\theta,$$
$$y' = x \sin\theta + y \cos\theta$$

about the origin and a translation (i.e., a specified amount of sliding in a specified direction)

$$x' = x + a, \qquad y' = y + b.$$

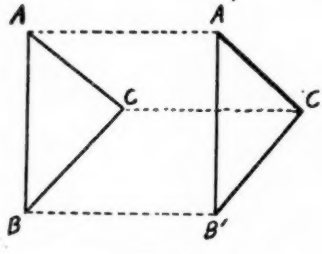

Fig. 3

For example, in Figure 3 we may think of triangle $A'B'C'$ as obtained by a translation of triangle ABC. In Figure 4 we may think of triangle $A'B'C'$ as obtained by a rotation of triangle ABC about the point O. Under each of these transformations (i.e., under translations and rotations) we find that size, shape, area, and magnitudes of angles are unchanged or invariant. Then since any rigid motion on the plane may be expressed in terms of translations and rotations, the above properties

† See B. E. Meserve, "Using Geometry in Teaching Algebra," The Mathematics Teacher, XLV (December, 1952), 567–71.

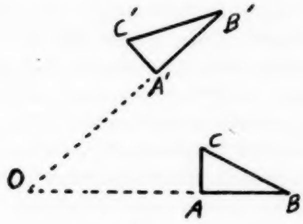

Fig. 4

are invariant under all rigid motions, that is, under all Euclidean transformations.

Let us now consider a transformation that is not a rigid motion. For example, let us consider a uniform stretching or, as it is technically called, a dilation such as that given by the equations

$$x' = 2x, \qquad y' = 2y.$$

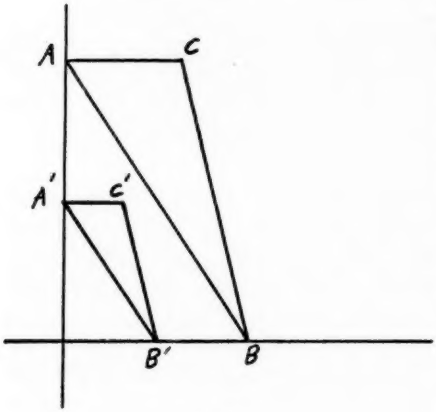

Fig. 5

Under this transformation every point on the plane is moved twice as far from the origin as it was originally. As in Figure 5, we find that under any dilation continuity, the shape of figures, and the magnitudes of angles are unchanged (invariant) but size, lengths of segments, and areas are changed (are not invariant).

In topology continuity is the only underlying invariant. The assumption of continuity may be formally stated in several ways. With a few unnecessary limitations due to the nature of the space in which we live, we may visualize the assumption of continuity as implying simply that there can be no cutting, no tearing apart and no folding together. If two points are joined by a curve, that curve cannot be cut. If a point is on a curve, it must remain on that curve even though the curve may be twisted, stretched and distorted in many ways.

Try to imagine a geometry in which size and shape are unimportant. For example, consider a curve as represented by a very elastic string. If, as in Figure 6, the ends of the string are not joined and the string does not cross itself, we call the curve a simple open curve. The string may be continuously transformed, that is without any cutting or folding together, into each of the curves represented in Figure 6. In topology these curves are equivalent since any one may be made into any other by a topological, that is a continuous, transformation. They are equivalent in topology in the same sense that two congruent triangles are equivalent in Euclidean geometry. Size and shape have no meaning in topology but there are still some underlying common properties—those that depend only upon continuity.

The central figure in Figure 6 may be

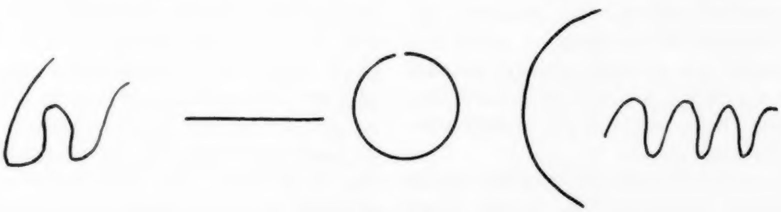

Fig. 6

considered as a circle with a segment missing. It is not equivalent to a circle since one is not allowed to cut the circle or remove a segment. A circle is a simple closed curve, i.e., a curve such that one can start at any point, traverse the complete curve and return to the starting point without passing through any point twice and without jumping from one arc of the curve to another. In general any simple closed curve may be visualized as a very elastic string with its ends joined and which does not cross itself. Such a string or curve may be continuously transformed into a circle, a square, a trapezoid, or any convex polygon. It may be transformed into any of the shapes in Figure 7.

have in mind may be stated very simply: Any simple closed curve has an inside and an outside. The circle, the gingerbread boy, and the other curves in Figure 7 have an inside and an outside. This property may be emphasized by coloring or shading the inside or bounded region of the curves. In the case of the right-hand figure in Figure 7 the inside may be found by starting at an exterior point and crossing the curve just once. The designation of the bounded region as the inside is conventional but not strictly necessary. Indeed, we have probably all heard the story of the drunk going around and around a large tree crying, "Let me out," "Let me out."

In advanced mathematics we say that any simple closed curve in a plane divides the plane into two regions. This is the Jordan Curve Theorem. The curve is the common boundary of the two regions, and one cannot cross from one region to another without crossing the curve. The Jordan Curve Theorem is a very powerful theorem and yet a very simple theorem. It is independent of the size and shape of the curve. It is a topological theorem.

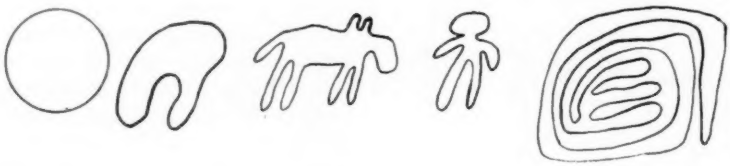

Fig. 7

I am sure that many children can do better art work than I have but I hope that you are sufficiently impressed by the variety of shapes of simple closed curves. In topology all of these curves are equivalent. Now in Euclidean geometry any two equivalent curves have the same shape, size and area. What do the curves in Figure 7 have in common? Shape, size, angles, straight lines are certainly not common properties. What is left to be considered? Let us remind ourselves that we are probing at the foundations of mathematics. The common property that we seek must be a fundamental property. Some of you may consider it a trivial property because of its simplicity. I prefer to believe that the really basic properties of mathematics are inherently simple.

Probably many of you have now found a common property of the simple closed curves in Figure 7. The property that I

How can such a simple theorem have any significance? It provides a basis for Euclid's assumption that any line segment joining the center of a circle to a point outside the circle must contain a point of the circle. It provides a basis for the existence of a zero of a polynomial on any interval on which the polynomial changes sign. It provides a basis for Venn diagrams in any two-valued logic. For example, true statements may correspond to points inside the curve, false statements to points

outside the curve. This simple theorem regarding the existence of an inside and an outside of any simple closed curve may also be used to answer questions raised by a problem on which most of you have probably spent many hours.

Consider three houses in a row and three other objects in a second row. The problem is to join each house to each of three objects by an arc in such a way that no two arcs cross. Several years ago I tried this problem when three houses and three utilities were involved. Tucker and Bailey* consider three houses, a haystack, a well and a dovecote. Whatever the objects are called, the problem is equivalent to joining each x to each o in Figure 8. It

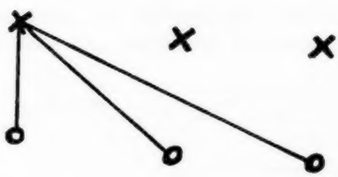

FIG. 8

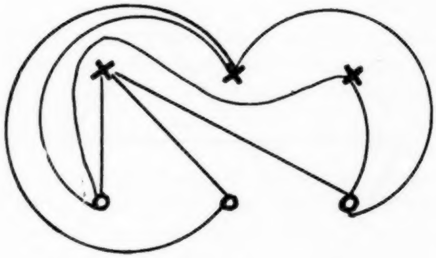

FIG. 9

is easy to designate paths from one house, say the x on the left, to each of the three objects. One can also designate the three paths from the second house as in Figure 9. Then one can designate two of the paths from the third house as in Figure 9 but it is not possible on a Euclidean plane to draw the path from the third house to the remaining object below. This assertion

* A. W. Tucker and H. S. Bailey, Jr., "Topology," *Scientific American*, CLXXXII, 18–24

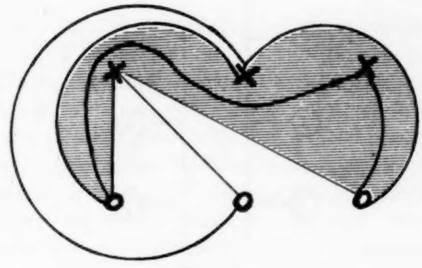

FIG. 10

is based upon the fact that the simple closed curve indicated in Figure 10 divides the plane into two regions, the third house is inside the curve (shaded area), the remaining object is outside and the two cannot be joined without crossing the curve.

Another problem that is based upon the properties of a simple closed curve may be found in *Mathematics and the Imagination* by Kasner and Newman. A Persian Caliph reportedly endeavored to select the best suitor for his daughter by posing two problems. The problems appear very similar at first sight. The suitors were asked to draw arcs joining the corresponding numbers in each of two figures (Fig. 11 and Fig. 12) in such a way that the arcs do not intersect with each other or the given figures. In the case of Figure 11 the problem is trivial (Fig. 13). However, we can be confident that the Caliph's daughter died an old maid if her father insisted upon a solution of the second problem. The first and second pairs of numbers may be joined but the third pair

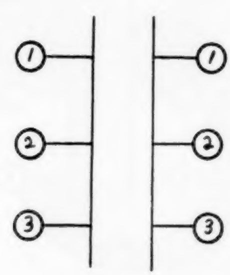

FIG. 11

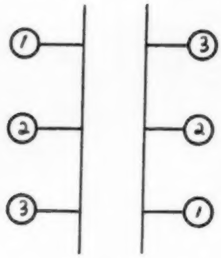

FIG. 12

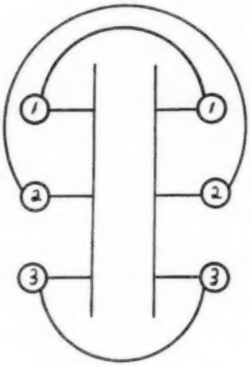

FIG. 13

cannot be joined. This assertion is also based upon the properties of a simple closed curve as indicated in Figure 14. One element of the third pair is inside the curve (shaded area) and the other is outside. By the Jordan Curve Theorem the two points cannot be joined in the plane without crossing the curve.

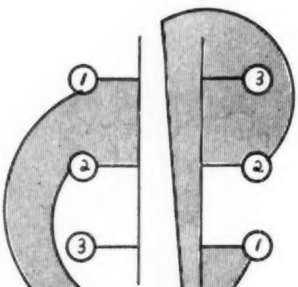

FIG. 14

Let us now turn back from our applications of the properties of simple closed curves to topologically equivalent figures. We have seen that all simple open curves (Fig. 6) are topologically equivalent in that any one can be continuously transformed into any other one. Similarly all simple closed curves (Fig. 7) are topologically equivalent. We may also consider curves (Fig. 15) that are equivalent to a figure eight or curves equivalent to more complicated curves. Indeed the study of knots forms a part of topology.

There also exist topologically equivalent surfaces corresponding to the sets of equivalent curves that we have considered. For example, many surfaces are topologically equivalent to a sphere. In particular, all convex polyhedrons, all surfaces of solids bounded by planes and such that

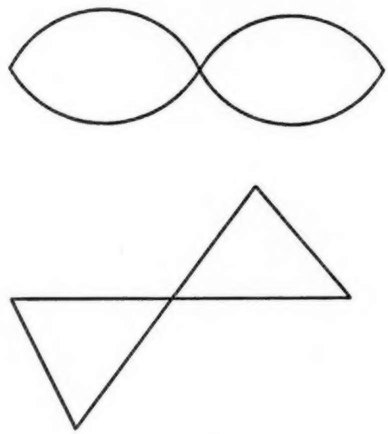

FIG. 15

any line segment joining points of the surface contains only points of the solid, are topologically equivalent to a sphere. We also consider spheres with handles. A sphere with one handle (Fig. 16) may take on many forms. If we shrink the sphere into the handle we obtain a doughnut as indicated in the sequence of figures in Figure 17. If the sphere is visualized as a tennis ball without enough air in it, so that the top may be pushed in, the

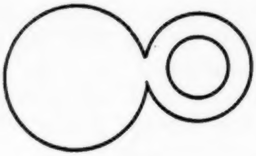

FIG. 16

FIG. 17

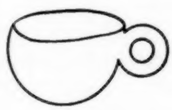

FIG. 18

sphere with one handle may be continuously transformed into a teacup (Fig. 18). We may also make the sphere with one handle into a builders block with one hole in it or a single tile. One of my students once told me that a doughnut was topologically equivalent to a flower pot. For a moment I was worried. Then I remembered the drain hole in the bottom of the pot. The drain hole is necessary. Flower pots, single tiles, doughnuts, teacups are all topologically equivalent. We may spend many pleasant hours considering topological invariances but let us now turn to consequences of topological invariances, problems and properties based upon these invariances. There are many elementary and many advanced topological problems. For example, spheres with handles may be associated with the integrability of algebraic functions. Such problems seem far removed from my topic of topology for secondary schools. However, coloring maps should appeal to everyone.

A preschool child will frequently color a whole sheet of paper a single color. It is possible to color a map of a single country with one color. In general, two countries are colored different colors if they have a common arc on their boundary. If they have only single points on their common boundary, they do not need to be colored different colors. We may color a green island in a blue ocean with two colors. Similarly, we may construct a map that may be colored with three colors. For example, we may consider an island divided into two countries as in Figure 19. We have now seen that maps may be colored using one, two or three colors. Furthermore, if a country has an odd number of neighbors, as in Figure 20, four colors are needed. In the map of the United States the state of Kentucky has an odd number of neighbors. One color, two colors, three colors, four—do we ever need more? This is an open question. No one knows the answer. However, before you start assigning this problem to pupils in elementary school, I should mention that four colors are sufficient for maps containing up to 83 countries. Thus any maps requiring more than four colors, if such exist, must have more than 83 countries. This problem is independent of the size and shape of the countries. It is a topological problem.

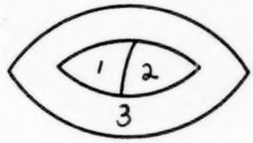

FIG. 19

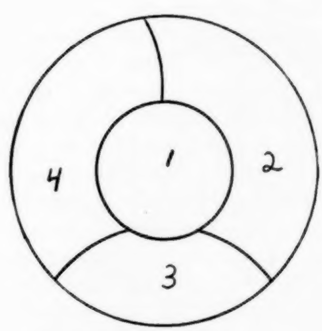

FIG. 20

Another famous topological problem is concerned with the bridges in the city of Koenigsberg. There was a river flowing through the city, two islands in the river and seven bridges as in Figure 21. The people of Koenigsberg loved a Sunday stroll and thought it would be nice to take a walk and cross each bridge exactly once. But no matter where they started or what route they tried, they could not cross each bridge exactly once. This caused considerable discussion. Gradually it was observed that the basic problem was concerned with paths between the two sides of the river A, B and the two islands C, D as in Figure 22. With this geometric representation of the problem it was no longer necessary to discuss the problem in terms of walking across the bridges. Instead one could discuss whether or not the curve (Fig. 22) associated with the problem is traversable, i.e., whether or not one could start at some point of the curve and traverse each arc exactly once.

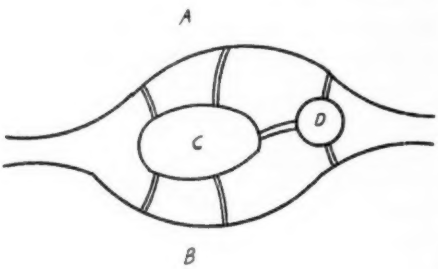

Fig. 21

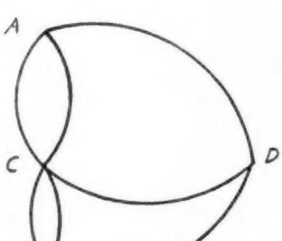

Fig. 22

The curve is often called the graph of the problem. In this form the problem could be considered by people who had never even been to Koenigsberg. The problem was solvable if and only if its graph was traversable.

When is a graph traversable in a single trip? One can walk around any city block and it is not necessary to start at any particular point. In general, one may traverse any simple closed curve in a single trip. This may surprise some children in the case of figures like the one at the right in Figure 7 but it is a basic property of all simple closed curves. We next consider walking around two blocks and down the street separating them (Fig. 23). This problem is a bit more interesting in that it is necessary to start at B or E. Furthermore, if one starts at B one ends at E and conversely. Note that it is permissible to pass through a vertex several times but one can only traverse an arc once. The peculiar property of the vertices B and E is based upon the fact that there are three arcs terminating at each of these vertices whereas the other vertices are each on two arcs. A similar observation led a famous mathematician by the name of Euler to devise a complete theory for traversable graphs.

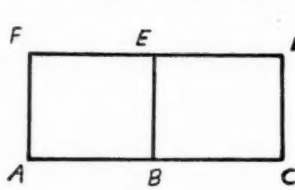

Fig. 23

Euler classified the vertices of a graph as odd or even. A vertex that is on an odd number of arcs is called an odd vertex; a vertex that is on an even number of arcs is called an even vertex. Since every arc has two ends there must be an even number of odd vertices in any graph. Any graph or network that has only even vertices is traversable and the trip may

be started at any vertex. Furthermore the trip will terminate at its starting point. If a graph contains two odd vertices, it is traversable but the trip must start at one of the odd vertices. It will then terminate at the second odd vertex. If a graph has more than two odd vertices, it is not traversable in a single trip. In general, a graph with $2k$ odd vertices may be traversed in k distinct trips. The graph in Figure 22 has four odd vertices. It cannot be traversed in a single trip and the famous problem of the seven Koenigsberg bridges does not have a solution in our geometry. The discussion of this problem in the article by Tucker and Bailey mentioned above is concluded with the statement that Tucker had actually walked across each of the bridges exactly once in 1935. There were eight bridges at that time.

Frequently we see in advanced mathematical theories only complicated manipulations and intricate statements involving precisely worded definitions and theorems. It is refreshing as well as enlightening to look back occasionally at the roots of the theory and see the problems that started great minds working for generalizations that have led to present theories. The Koenigsberg bridge problem is independent of the size and shape of the objects under consideration. It is a topological problem. It has in fact been considered by some writers to be the starting point of the theory of topology.

Let us conclude our consideration of topology with a few comments upon a surface that has several very unusual properties. This surface is one sided. A fly can walk from any point on it to any other point without crossing an edge. Unlike a table top or a wall, it does not have a top and a bottom or a front and a back. This surface is called a Moebius strip and may be very easily constructed from a rectangular piece of paper such as a strip of gummed tape. Size is theoretically unimportant but a strip an inch or two wide and about a foot long is easy to handle. We may construct a Moebius strip by twisting the strip of gummed tape just enough to stick the gummed edge of one end to the gummed edge of the other end (Fig. 24). If we cut across this strip we again get a single strip similar to the one we started with (Fig. 24). But if we start with a rectangular strip and cut around the center of the Moebius strip (see the dotted line in Fig. 25) we do not get two strips. Rather we get one strip with two twists in it. William Hazlett Upson used this peculiar property of Moebius strips in his story of "Paul Bunyan and the Conveyer Belt" (*Ford Times*, July 1949, pp. 14–17). Anyone can use it with children and adults.

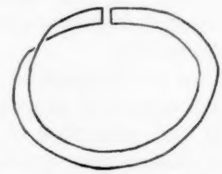

FIG. 24

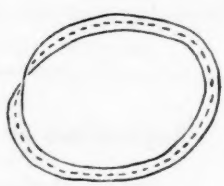

FIG. 25

I have used these one-sided surfaces as place cards at a seven-year-old's birthday party. They are suitably shaped for this purpose. Then while waiting for dessert the youngsters were encouraged to cut the strip down the middle while guessing what the result would be. They were suitably impressed when they found only one piece and were anxious to cut it again. Once more they were suitably impressed when they found two pieces linked together. Almost a year later one of the boys asked me about the piece of paper that was in only one piece after it was cut in two. The smaller children can usually

cut the strip at most twice, but older children and adults enjoy cutting the strip several times and several ways to see what will happen. Children and adults will ask questions that the teacher cannot answer and most college mathematics professors cannot answer. This will be good for all concerned since it will impress upon them that there is more to mathematics than formal algebraic manipulations and classical geometric constructions.

All teachers of mathematics can use this and other topological properties to challenge and interest children of all ages as well as adults. These properties may be used with superior students, mathematics clubs and for the general enrichment of one's teaching. They will help us encourage a genuine interest in fundamental principles of mathematics.

I have tried to illustrate for you some of the consequences of continuity—that is, some of the properties of topology, the geometry based upon continuous transformations. I hope that you have gained a new appreciation for Euclidean geometry in addition to recognizing that there exist other geometries. In particular there is topology, a geometry which has Euclidean geometry as a special case and which may be used, even in elementary schools, to interest and challenge the students and to emphasize the importance of one of the fundamental concepts of mathematics—continuity.

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Registrations at Annual Joint Meeting with the NEA of the National Council of Teachers of Mathematics

Miami Beach, Florida, June 29, 1953

State	*Total*
Alabama	10
California	5
Colorado	1
Connecticut	1
District of Columbia	3
Florida	71
Georgia	7
Illinois	3
Indiana	5
Iowa	1
Maryland	2
Massachusetts	1
Michigan	5
Minnesota	2
Missouri	2
Montana	1
Nebraska	2
Nevada	1
New Hampshire	1
New Jersey	1
New York	2
North Carolina	1
Ohio	7
Oklahoma	1
Oregon	1
Pennsylvania	10
South Carolina	2
South Dakota	1
Tennessee	3
Texas	2
Virginia	5
Washington	2
West Virginia	6
Wisconsin	3
Unclassified	3
TOTALS	174

A Unit on the History of Arithmetic

(for Seventh or Eighth Grade)

By MAY L. WILT

Instructor, College of Education, West Virginia University

MANY TEACHERS say that the best way to teach the history of mathematics is to introduce it along with the development of mathematical content. This point of view is logical, but it has two inherent weaknesses:

1. When pressed for time, most teachers omit teaching the history.

2. Unless an over-all school effort exists, the teaching of the history of mathematics lacks unity and results in an *overemphasis* of some phases and an *underemphasis* of others.

In order to overcome these two weaknesses, the University High School of West Virginia University is experimenting with the introduction of a unit on the history of arithmetic in its seventh and eighth grade mathematics classes. Frequently this unit is used as an introduction to the work for the year.

"LET'S BE NUMBER SLEUTHHOUNDS"

Aims: To help you:

1. Realize how early man developed numbers and numerals.
2. Study several different number systems.
3. Understand the story of your own number system.
4. Learn how to check computation.

Introduction: Until he was faced with a shortage of food, clothing, shelter, or manpower, man had little need for counting or measuring. Today he is bombarded with problems involving numbers and numerals: taxes, insurance, war debts, the high cost of food, school expenses, installment buying, paying for a house, Social Security, etc., etc. The solution of these problems helps spell financial success to an individual, to a family, or to a nation.

*Bibliography:** You will enjoy reading in the following books:

1. American Council of Education, *The Story of Numbers.*
2. American Council of Education, *The Story of Weights and Measures.*
3. American Council of Education, *The Story of Writing.*
4. American Council of Education, *The Story of Our Calendar.*
5. Bakst, Aaron, *Mathematics, Its Magic and Its Mastery.*
6. Bendick, Jeanne, *How Much and How Many.*
7. Burroughs Adding Machine Company, *Fascinating Figure Puzzles.*
8. Burroughs Adding Machine Company, *The Story of Figures.*
9. Cajori, F., *History of Arithmetic.*
10. Ford Motor Company, *He Measured in Millionths.*
11. Ford Motor Company, *How Long Is a Rod?*
12. Karpinski, L. C., *The History of Arithmetic.*
13. Marchant Calculating Company, *From Og to Google.*
14. Smith, D. E., *Number Stories of Long Ago.*
15. Smith, D. E., *The Wonderful Wonders of One—Two—Three.*
16. Smith, Ginsburg, *Numbers and Numerals.*
17. Wiese, Kurt, *You Can Write Chinese.*
18. Encyclopedias.

As you read, keep the following points in mind:

1. The origin of number names.

* Complete bibliography is provided at end of the unit.

2. The base of each number system.
3. The history of some everyday mathematical words.
4. Words used to denote uncounted groups.
5. Different types of counters.
6. Counting boards.
7. Various ways people have added, subtracted, multiplied, and divided.
8. How early people handled fractions.
9. How to build a number system.

You may be able to suggest other points.

Interesting Activities: If you have read widely, you will be able to do the following:

1. Relate the story or meaning back of each of these words:

digit	November	score
tally	cuneiform	check
counter	hieroglyphics	fraction
calculate	rod	cent
line	foot	September
dollar	yard	"so many hands"
October	December	
mile	span	

2. Make a list of words you use today for uncounted groups.
3. Make up number names and count from 1 to 20.
4. Suppose the only numerals you had were 1, 2, 3, 4, 5, and 0, show how to write each of these: 6, 12, 15, and 20.
5. Make the following:
   a. A clay tablet.
   b. A wax tablet.
   c. A roll of papyrus.
   d. An abacus.
6. Certain numbers seem to have been used more than others. Following are some examples of the use of number three suggested by Dr. Grossnickle. Add other groups of three.
   a. Three Blind Mice
   b. Three Cheers
   c. The Three Little Pigs
   d. Tom, Dick, and Harry
   e. Three Ages of Man
   f. Morning, Noon, and Night
   g. Father, Mother, Child
   h. Sun, Moon, and Stars
   i. Three Square Meals
   j. Earth, Air, and Water

Writing Numerals: The world has known hundreds of number systems, but you will perhaps work with five different ones.

Activities:

1. Become familiar with the following systems, noting symbols, bases, peculiarities, and operations:
   a. Egyptian
   b. Babylonian
   c. Chinese stick
   d. Roman
   e. Hindu-Arabic
2. In several different systems, write the numerals from 1 to 100, your age, the year 1953, and any other numbers that may interest you.
3. Using match sticks, make a poster showing how the Chinese stick system operates.
4. Learn how to multiply by the process of doubling.
5. Take a system other than your own, and use it to add, subtract, multiply, and divide.

Early calculating devices: Early man kept records on his fingers and toes. This accounts for the use of 5, 10, and 20 for number bases. It also accounts for the use of the term "digit." Look up the following:

1. Fingers and toes as counters.
2. Pebbles as counters.
3. Notched sticks and tallies. Explain how they were used.
4. The abacus. Explain its development, how it operates, why it delayed the need for zero, and the various forms of the abacus now in use throughout the world.

How Early Records Were Preserved: Before the invention of paper, records were kept on stone, clay tablets, papyrus, skins of animals, palm leaves, bark of

trees, and wax tablets. Explain why the materials for writing varied from country to country.

Then you will want to do these:

1. Look up the origin of paper.
2. Make an exhibit showing how people preserved records (using the appropriate number system for each specific type).

Checking Computation: Computation must be accurate if it is to be of value. Learn how to check by each of the following methods:

1. An abacus.
2. Naperian rods.
3. Casting out "nines."
4. Any other method you have found convenient.

Tricks and Number Magic: Mathematics is filled with interesting tricks and puzzles. You will enjoy making a collection of the ones that appeal to you and then presenting your findings to the class.

Conclusion: After careful preparation, write an article on two or three of the following topics:

1. "My Number System and Why I Should Know How To Use It Accurately."
2. "Numerology."
3. "The Story of the Calendar."
4. "The Two Types of Egyptian Numerals."
5. "How Nature Uses Numbers."
6. "How Numbers Got Their Names."
7. "Fun with the Number Nine."
8. "Magic Squares and Circles."
9. "Number Names Beyond Billions."
10. "The Number System of the Maya Indians."
11. "The History of Zero."
12. "My Own Number System."
13. "Number Names in Several Different Languages."
14. "Counting with Pebbles."
15. "Arithmetical Words I Should Know."

The above articles may be shared with the group.

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HAVE YOU READ?

Euler, Leonhard, "Leonhard Euler and the Koenigsberg Bridges," *Scientific American*, July 1953, page 66.

The problem of crossing but not recrossing the seven bridges of the river Pergel which confronted the evening strollers of Koenigsberg inspired Euler to consider the possibilities of that great branch of mathematics called topology. You will want to read this direct translation of his thinking as he set it down while proving the problem unsolvable. Not only is the problem of interest, but his clearness of demonstration, approach to problems, specificity of definition, and order of presentation is something to behold. The spark of the scientist shines brightly as we follow his thinking on beyond the solution of a particular problem into the generalization.—Phillip Peak, Indiana University, Bloomington, Indiana.

The Carpenter's Rule: An Aid in Teaching Geometry

By Ethel L. Moore
Torrington High School, Torrington, Wyoming

The carpenter's rule is an excellent manipulative device to use in teaching meaningful informal and formal geometry. A carpenter's rule (folding or zig-zag) is a 72-inch or 84-inch measuring stick that folds at joints 6 inches apart.

There should be several 72-inch rules for the teacher and pupils to use in demonstrations, and each pupil should have, for his use in class, a rule of at least 36 inches. The wooden and metal carpenter's rules are more easily disjointed at the 36-inch mark than are the plastic ones. On many occasions it will be advantageous for the pupils to work in small groups.

Some concrete uses where the rule may be effective follow:

Lines, the kinds of, and their positions and relations to each other can readily be taught with the carpenter's rule. Lines can be extended, added, and subtracted, and the line segment can be demonstrated. The plastic rule is pliable and can be used to represent a curved line to illustrate that a straight line is the shortest distance between two points.

Angles become "alive" when a pupil builds an angle by folding the carpenter's rule at the mid-point and rotating one section of the rule. Lengthen and shorten the sides of the angle to illustrate that the size of the angle is independent of the length of its sides. By manipulating the rule, make the various kinds of angles and show their relations to one another. By working in groups, build the parallel lines and cut them by a transversal to demonstrate equal angles. Also show that these equalities are false when the lines are non-parallel.

Triangles of all kinds are made by the pupils to demonstrate many geometric facts. By experimenting, pupils learn that the sum of the two sides of a triangle must be greater than the third side; that the longest side is opposite the largest angle; and that a triangle is a rigid figure. Altitudes and medians can be shown on all three types of triangles by using rulers and pencils to supplement the carpenter's rule. Form triangles and show the perimeter by "unforming" it. The three congruent triangle theorems can be demonstrated and realized by pupils working in groups.

Quadrilaterals of all kinds can be made. However, first have all the pupils make rectangles using the six sections. Call attention to the altitude, the equality of the angles, the sum of the four angles, the sum of any two consecutive angles, the parallel and equal opposite sides, and the two diagonals. Skew the rectangle to form the parallelogram and demonstrate various truths regarding it. Demonstrate the change of the area of the parallelogram as the altitude diminishes. The square and rhombus can be treated in the same way.

After demonstrating the perimeter and area of a parallelogram, teach the area of a triangle and trapezoid by building in the diagonals. Build the various polygons, pentagon, hexagon, octagon, decagon, and duo-decagon. Develop with the pupils, working in small groups, that Σ interior $\angle$s of a polygon $=(n-2)$ $180°$ and that Σ exterior $\angle$s of a polygon $=360°$. Use the carpenter's rule to teach the variance in the area of plane polygons when the base is doubled or trebled and the altitude remains unchanged, when both are uniformly increased.

The above suggestions are only a few of the many meaningful uses of the carpenter's rule in teaching geometry. Pupils like using the ruler and make many discoveries for themselves.

Comments on Computation with Approximate Numbers*

By CECIL B. READ
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THERE ARE several excellent articles dealing with computation with approximate numbers. These vary from a very elementary treatment to rigorous proofs of the principles involved. It is assumed that the general ideas are rather well-known. For example, computation cannot increase the accuracy of the original data. Again there is distinction between the precision and accuracy of a measurement. The precision of an approximate number is determined by the size of the unit of measurement. The accuracy depends upon the number of significant digits involved.

It is not always the size of the error which is important, but the relative error. An error of one foot would be very large in attempting to fit a door to an opening. An error of only one foot in determining the distance from Lincoln to Wichita would mean an extremely small relative error. It is emphasized that the accuracy of a number depends upon the number of significant digits involved, not the position of the decimal point.

We often find such statements as this: The measurements 3456 inches, 34.56 inches and 0.3456 inches, are all equally accurate, since they have the same number of significant figures, but the last measure is the most precise, since it has a smaller or more precise unit of measurement. This statement is correct, but sometimes from this a false assumption arises. If I measure something in the metric system and find it is 234 millimeters this can also be written as 23.4 centimeters, or 0.234 meters. These measurements are all equally accurate and *also* have equal precision. This last fact is often misunderstood. The same unit of measurement has been *expressed* in terms of different units.

It is sometimes stated that if indivisible units are counted the number is said to be exact. This is not necessarily true. If it is reported that the attendance at a football game is 14,328, this may have a spurious exactness. Are we sure that two people did not pass through the turn-stile at the same time? Are we sure that we have counted all ushers, hot dog vendors, and the like?

Anyone who has dealt with computation of approximate numbers realizes that the question of significant digits is vital. There seems to be little trouble in determining what are the significant digits, except in the case where the digit 0 is involved. Zero is always significant when it appears at the right of a significant digit in a decimal fraction. It is always significant when it appears between other digits. The problem arises when one or more zeros appear at the end of a number or when they merely place the decimal point. Much of the difficulty can be avoided if we use scientific notation. If the distance from the earth to the sun is given as 93,600,000 miles we do not know the accuracy. If we write it 9.36×10^7, the meaning is clear. There are exactly three significant digits. If we write it 9.360×1 there are four significant digits.

Without the use of the scientific notation it is usually not possible to determine whether the number 100 is correct to one, two, or three significant digits. Some have advocated a special symbol for a zero which merely locates the decimal point, as contrasted with one which is a significant digit. We could, then, by writing 10Ø indicate a number correct to two significant digits. Actually, there seems little

* Presented at the Thirteenth Christmas Meeting of the NCTM, December 30, 1952, at Lincoln, Nebraska.

need for such a symbol since scientific notation solves the difficulty by writing 1.0×10^2.

A similar question arises with an angle of 37°20′—is this correct to the nearest minute or the nearest ten minutes? Presumably, the question might be made clear by writing 37° $(2.0\times 10)'$ but certainly this notation is not in common use. In the complete absence of any information, it may be necessary to state what assumption is used in arriving at an answer involving some trigonometric function of 37°20′. One might give two answers, one on the assumption of approximation to the nearest ten minutes, the other on the assumption of approximation to the nearest minute.

Almost all discussions of approximate computation assume that approximate numbers have an error not in excess of 0.5 times the unit of measure. It must be pointed out that this is not always true. The physicist may speak of a doubtful figure, and there could be circumstances where a measurement reported as 87.6 is somewhere between 87.3 and 87.9. In this discussion, other than mentioning this fact, the customary assumptions will be made.

Most discussions of approximate numbers mention the process of *rounding*. There is general agreement that if the first digit to be dropped is less than 5, the last digit retained is not changed. If the first digit dropped is greater than 5, the last digit retained is increased by unity. If the digit dropped is exactly 5, authorities disagree. Some give the rule that in this case the first digit retained is increased by unity. Others, using what is called the even number rule, state that we round so the result shall be an even number, the underlying theory being that this will increase the result in about half the actual cases. Suppose we apply these rules in a problem in interpolation: Required log 2.4625, using a five place table:

$$\log 2.4620 = 0.39129$$
$$\log 2.4630 = 0.39146.$$

Half the difference, 0.00017, is 0.000085. If we round by the rule that dropping a 5, we increase the digit retained, the required logarithm is 0.39138. If we use the even number rule, rounding 0.000085 to 0.00008, then adding, the result is 0.39137. If we use the even number rule, adding the correction, then rounding, the result is 0.39138. Rules will not give unique results unless clearly stated. (The fact that the value, as obtained by rounding from a seven place table, is 0.39138, does not prove one method superior.) Some authorities round first, then add; others add, then round to make the *final result* even.

Interpolation may be used to illustrate approximate numbers. A student sometimes thinks it easier to locate a five place table than to learn interpolation in a four place table. He may be surprised to find that rounding from a five place table gives $\log\ 2.631 = 0.4201$, while interpolation using a four place table yields 0.4202. Ask him whether this involves approximate numbers.

If we are dealing with exact numbers, technically it is correct to carry as many places as can be obtained in the operation. However, it may not be practical or expedient to report results to a large number of digits. In computations based on approximate numbers, the results are valid only for a specified number of digits.

In elementary work, we may be more concerned with a workable rule than with one theoretically correct. It is relatively rare to find an addition problem involving more than five or six approximate numbers. If we keep in mind that computation does not increase the accuracy of the original data, and that all measurements to be added or subtracted should be measured to the same units, a working rule would be that in addition or subtraction, we first round all numbers to the precision of the least precise number.

In multiplication or in division with approximate numbers, the general rule is that the result cannot be relied upon for more than n significant digits where n is

the number of significant digits in the approximate number with the fewer number of significant digits. Unfortunately, this rule is often misunderstood to mean that the result will be correct to n significant digits. The result *may be* correct with this accuracy, we certainly cannot depend on greater accuracy, but there is no assurance that there *will be* this accuracy. For illustration, consider the product of the approximate numbers 864.7 and 4.3. The least value the product could have is $864.65 \times 4.25 = 3674.7625$; the greatest value is $864.75 \times 4.35 = 3761.6625$. These results do not even agree in the second digit, certainly no following digits are accurate. The rule says we cannot rely on *more than* two significant digits. We may write the answer 3.7×10^3.

One rule which has merit is that if the numbers involved do not have the same number of significant digits, we first round the more accurate number until it contains one more significant digit than the other. After the computation is carried out until we have at least one more significant digit in the result than in the least accurate number, we round the final result to the same number of significant digits as were in the least accurate number. This may save some work in multiplication, and a considerable amount of work in division. However, with modern computing machines, more time may be spent in rounding the more accurate number than in carrying out the originally indicated calculation.

Once more it is emphasized that the usual rules do not necessarily give n significant digits where the least accurate number is correct to n significant digits. The result is reasonable; on the average it is more likely to be correct to n digits than not, but we cannot be certain. If we must be absolutely definite, we will have to state within what limits the result lies—for example, state a least possible product, and a greatest possible product.

Where three or more approximate numbers are involved, results which seem inconsistent may be obtained, depending on the manner in which the various numbers are combined. One working rule is that if possible, we combine the more accurate factors first, and that in partial results we retain one more digit than we will retain to the final answer (a few authorities suggest retaining two more digits). The important thing to remember is that with any rule, results will not be accurate to more than a specified number of digits, and *may not* be accurate this far. Hence, there is no need to be unduly perturbed by the fact that one method yields 23.6, another order of operations yields 23.5. If essential, we could use the greatest possible and least possible results. In a combined multiplication and division, this means that to get the greatest possible result we would need to use the greatest possible value of all factors in the numerator of the fraction, and the least possible value for all factors in the denominator; reversing for the least possible result.

Students and teachers still find certain difficulties arising. For example, if we seek the area of a rectangle $3'$ by $5'$, should we round to 20 sq. ft.? At first thought, the answer is yes, an answer which seems inconsistent, since 3.5×5.5 (the greatest possible product) is only 19.25. Two facts should be kept in mind: The product can not be relied on to more than one significant digit, and *may not* be this accurate. Secondly, the 20 is ambiguous—a better result would be 2×10^1 sq. ft. This result, implying some value between 15 and 25 square feet, does not seem as inconsistent.

What about the product of $\frac{1}{2}$ (15) (34)? One fundamental question arises—is the fraction $\frac{1}{2}$ an exact, or an approximate number? If exact, as for example, in the area of a triangle, we retain two significant figures in the product. On the other hand, a problem involving $7\frac{1}{2}$ gallons to the cubic foot is an approximation correct to two significant digits (a more accurate value is 7.481). In spite of this, textbooks will ask the number of gallons in a lake 100

feet deep, 70 miles long, 30 miles wide, allowing $7\frac{1}{2}$ gallons to the cubic foot. The result is likely to be expressed to eight or nine digits. One author, questioned about this practice, pointed out a brief statement to the effect that in the text, all numbers were to be considered exact. This raises two questions: Why, then, devote seven pages to discussing rounding, approximate numbers, etc.? And again, if numbers such as 18 feet are exact, what method of measurement was used? There would be no objection to using a value such as 18.000 feet.

When operating with logarithms, it may be well to avoid the use of the words significant digits as they relate to the mantissa of a logarithm. Probably it will be sufficient for most purposes to inspect a table of mantissas of common logarithms, and note that roughly, as the final digit of a three digit number changes, the third digit of the mantissa changes, etc. In other words, we may use a three place logarithm table (or a slide rule) if data are given to three significant digits, a four place table with data to four significant digits. Rather obviously, this rough rule will not hold throughout the table. Certainly it is absurd to attempt to use a different number of places for the mantissa, for numbers starting with the digit 1 or 2 than for those commencing with the digit 9.

One frequently finds statements about the number of significant digits to be used when trigonometric functions are involved. Based on the fact that trigonometric functions are ratios, and arise from division, we find it frequently stated without proof that angles measured to the nearest degree justify two significant digits; angles to the nearest minute justify four significant digits, to the nearest ten (or six) seconds justify five significant digits, to the nearest second justify six significant digits. Again it must be emphasized that these are general rules, they do not hold universally. As an extreme example, if $\sin x = 0.9999$, x might be anywhere between $89°0'28''$ and $89°25'37''$. This is far from agreement with the statement that angles measured to the nearest minute correspond to four significant digits. On the other hand, in the neighborhood of $1°$ a change in the fourth significant digit of the sine of the angle is frequently produced by a change in one second in the angle.

Books of a previous generation, and in too many cases, books of the present generation, have inexcusable statements and illustrations relative to computation. In many cases one studying the text would gain the impression that it is possible to measure exactly, and that all numbers and computations are absolutely exact. Perhaps in our attempt to correct this situation the pendulum may have swung a little too far. There is, and properly so, a considerable amount of attention being given to this subject at present. We must be careful, however, not to gain the impression that certain general rules will give unique or exact results. Rather, they give reasonable, but approximate results.

How much of this can be presented at the high school or junior college level? Certainly the student can learn something about the accuracy and precision of measurement as well as the accuracy of computation with approximate numbers.

Consultation of various references will reveal some disagreement about rules, although these disagreements are relatively minor. Likewise, these references will vary in the rigor of the treatment. Detailed proofs such as are found in the Twelfth Yearbook of the National Council of Teachers of Mathematics are valuable, probably the teacher will not present them to the class. Even with the simplest explanation, we should avoid the error of the students who take a home made transit, measure the elevation of the school flagpole, which they assume is perpendicular to the ground, which is likewise assumed horizontal (both probably in error), and calculate the height of the flagpole, carrying the result out to twelve digits.

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Algebra

Of late I've been scanning history's pages,
  But not for a reason you'd guess;
I've been delving away back through the ages
  To find out who started this mess.

On the ancient Egyptians I've placed some blame,
  They were among the first to do it,
Their learning no doubt has brought them fame—
  If I had my way, they'd rue it.

On the Arabians too, I vent my ire,
  They have done me a grievous wrong,
If to fame these people just had to aspire,
  Why not with verse or song?

And those Roman scholars who burned midnight oil,
  On them I would heap my wrath,
When I think of what they did, I boil!
  Why couldn't they have stayed in the bath?

Only the Greeks had the sense to know
  That the going was going to be rough,
They used rare judgment and just laid low—
  They didn't even touch the stuff.

Wise men from the East, men from the West,
  Italian, German, and Hindu,
All took part in the endless quest,
  What can *X* be made into?

Let's settle it now, between you and me,
  Let's settle it for worse or better,
I'm perfectly willing to let *X* be
  —Just the twenty-fourth letter

DORIS DASSAU*

* A second term pupil in elementary algebra at the Forest Hills High School, Forest Hills. New York.

Elementary Techniques in Maxima and Minima

By John A. Tierney
United States Naval Academy, Annapolis, Maryland

The application of the first derivative in the solution of maxima minima problems is an important topic in a first course in calculus. It is interesting to note that many of the standard textbook problems of this type can be solved without the use of the calculus and require only a knowledge of the mathematics customarily taught in the secondary schools. Whereas the calculus approach is usually quite straightforward, the elementary techniques which are effective vary considerably, some being trivial in nature and others requiring a certain amount of ingenuity.

An example of a trivial problem is that of maximizing the function

$$y=2/(1+x^2).$$

Most students can see by inspection that 2 is the maximum value y may assume. (Obviously, only real numbers are to be used in this paper.)

Extreme values of many algebraic and trigonometric expressions become evident after the expressions have been transformed slightly. As an illustration we consider the function

$$y=a \sin kx+b \cos kx$$

which has a wide range of applications. Employing one of the addition formulas from trigonometry, we rewrite the function in the form

$$y=(a^2+b^2)^{1/2} \sin (kx+\theta),$$

and see at once that the extreme values of y are $\pm(a^2+b^2)^{1/2}$.

We now consider a few problems whose solutions involve elementary geometry. One of the simplest which is found in most calculus texts is the following:

In Figure 1 a pumping station P is to be located on a straight road BC and is to pump water to points A and D located at distances BA and CD from BC. Find the position of P which makes the sum of the distances $AP+PD$ a minimum.

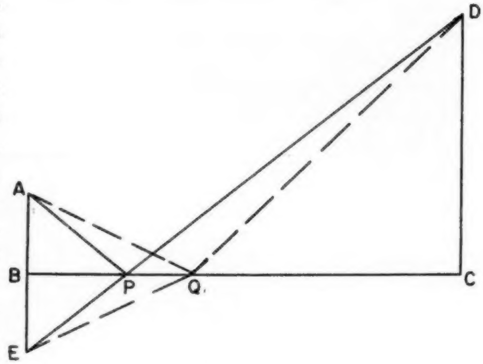

Fig. 1

If we extend AB to E such that $AB=BE$, then DE meets BC in the required point P, for if Q is a point on BC different from P, then

$$DQ+QE=DQ+QA>DE=DP+PA.$$

The next problem, in which the quantity to be minimized is an area, is also well known.

A spring is located a feet and b feet from two roads which meet at right angles. Find the path connecting the roads, passing by the spring, and cutting off a lot of minimum area.

In Figure 2 let $CS=a$ and $SA=b$. Construct $AB=a$ and draw line DSB. To prove that this is the required path draw an alternate path MSP as shown. Also construct a line through B parallel to OD and intersecting MSP at K. Since triangles MDS and KBS are congruent we see that the area cut off by MSP exceeds the area cut off by DSB, the excess

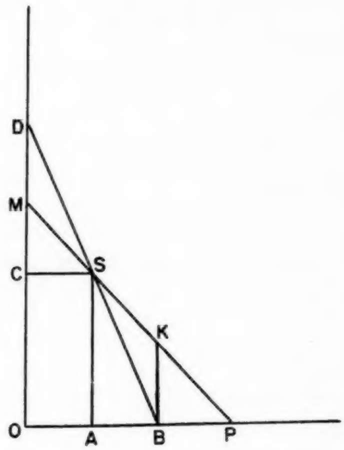

FIG. 2

being the area of triangle BKP. If point M is above point D a similar argument is employed.

All students of calculus will recognize the following problem in which the quantity to be maximized is the size of an angle.

A picture a feet in height hangs on a vertical wall with its lower edge b feet above the level of an observer's eye. How far from the wall should the observer stand in order that the angle subtended by the picture at his eye shall be a maximum?

In Figure 3, $AB=a$, $BC=b$, and CX is perpendicular to CA. Then the required distance CD is the mean-proportional between CB and CA and can be found by elementary algebra or geometry. To see this we note first that the circle passing through A, B, and D is tangent to CX at D. Now if P is any point to the right of C other than D, then

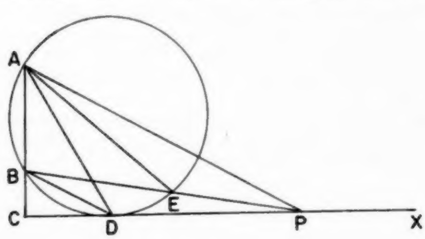

FIG. 3

$$\text{angle } BPA < \text{angle } BEA = \text{angle } BDA.$$

We observe that the above solution is unchanged if CA is not perpendicular to CX, a restriction which is made in the usual calculus solution.

We now consider a problem which permits a simple algebraic solution. The method used can be applied to numerous problems.

Towns B, C, and A are located 6 miles west, 6 miles east, and 10 miles south, respectively, of a point D. A road is to run north from A to a point P, and from P a branch is to run to B and another branch to C. Find the length of PC if the total length of road $l=AP+PC+PB$ is a minimum.

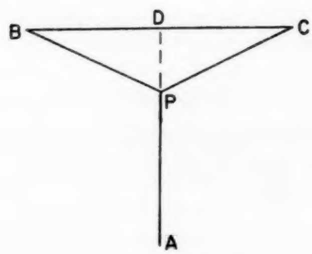

FIG. 4

In Figure 4 let $x=PC$ and we obtain

$$l=10-(x^2-36)^{1/2}+2x,$$

which, when solved for $3x$ in terms of l, yields

$$3x=2l-20\pm(l^2-20l-8)^{1/2}.$$

In order for x to be real, l must satisfy

$$l^2-20l-8\geqq 0.$$

The smallest positive value of l which will make x real is easily seen to be $l=10+6(3)^{1/2}$. This is the value we seek because it yields $x=4(3)^{1/2}$ which, being greater than 6 and less than $(136)^{1/2}$, is a permissible value of x. Thus when $x=4(3)^{1/2}$ miles, l is a minimum.

Our final problem can be solved geometrically or by the method just described but we will employ a somewhat different approach.

A man 6 miles from shore wishes to reach a point B on the shore 10 miles away. If he can row 2 miles per hour and walk 4 miles per hour, at what point P should he land in order to reach B in the least time?

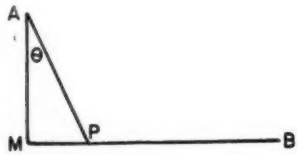

FIG. 5

Letting t denote the number of hours required to reach B we have from Figure 5

$$t=\frac{6 \sec \theta}{2}+\frac{8-6 \tan \theta}{4}$$
$$=3/2(2 \sec \theta-\tan \theta)+2.$$

We observe that t will assume its minimum value when $2 \sec \theta-\tan \theta$ takes on its least value.
Let

$$2 \sec \theta-\tan \theta=m.$$

Then

$$2-\sin \theta=m \cos \theta$$

or

$$2=\sin \theta+m \cos \theta.$$

Now let

$$\beta=\cos^{-1} (1+m^2)^{-1/2}$$

and we obtain

$$2=(1+m^2)^{1/2} \sin (\theta+\beta)$$

from which we see that m will be least when $\sin (\theta+\beta)=1$ or $\theta+\beta=\pi/2$. Then

$$m^2=3$$

and

$$\theta=\sin^{-1} (1+m^2)^{-1/2}=\sin^{-1} 1/2=\pi/6.$$

Finally

$$AP=6 \sec \theta=4(3)^{1/2} \text{ miles}$$

and $t=(3)^{3/2}/2+2$ hours will be required to reach B. (In general $\theta=\sin^{-1} r_1/r_2$ where r_1 is the rate rowing and r_2 is the rate walking.)

These are but a few of the various techniques which are applicable. It is hoped that the problems presented will be of interest to both secondary and college students. Some might enjoy finding other methods of solving the many maxima minima problems which are continually arising in mathematics.

HAVE YOU READ?

Brownell, William A., "The Effects of Practicing a Complex Arithmetical Skill Upon Proficiency in Its Constituent Skills," *The Journal of Educational Psychology*, February 1953, page 65.

Have you ever entered into a controversy with one of your colleagues over the question as to which should be done first, perfecting the fundamental skills of arithmetic before starting complex problems, or allowing the complex problem to provide practice in those skills? If so, you will want to read Brownell's paper on the subject. The results of this study are based on information taken from seventeen fifth-grade classes in the Chicago area.

Although his conclusions are interesting and valuable, the real worth of the article is the insight he gives the reader into the inherent complexities of teaching effectively a rather simple problem in long division.—PHILLIP PEAK, Indiana University, Bloomington, Indiana.

Ultra-Curricular Stimulation for the Superior Student

By Daniel B. Lloyd
Wilson Teachers College, Washington, D. C.

Adopting the hypothesis, rather well substantiated by research studies, that achievement is dependent on drives of interest and effort, the implication is clear for teachers to produce and maintain these drives. Research indicates no exclusive claim of mathematics to any specific abilities, therefore other fields of study attract, and afford satisfaction to, our potential mathematics students. It has also been well established that the most impressionable years of age are the formative ones during which the potential mathematics students are enrolled in the secondary school program.

In support of this we may recall that in 1902 the Swiss mathematical periodical *L'Enseignement Mathematique* undertook an inquiry into the working habits of mathematicians. Questionnaires were sent to a large number, of whom over a hundred replied. One question was: "At what age did mathematics first seize you?" Out of a total of 93 replies, 35 said before the age of 10; 43 said 11 to 15; 11 said 16 to 18; 3 said 19 to 20; and one lone laggard said 26. This important study focusses our attention on the fact that the most able students are rarely captured by mathematics after what are today normally their high school years; and that only during this crucial period, then or never, can we save them for our profession, rather than let them slip into other attractive fields for which they are probably as well suited. These facts point to the need for a sound and thoughtful approach to the problem of attracting and holding the interest and enthusiasm of the superior student in our field.

The particular phase of the subject to which I intend here to address my remarks deals with the problem of supplementing, for the sake of the superior student, the subject matter of the regular high school courses.

Two important tasks confront the teacher of superior students: (1) developing his enthusiasm for the subject and (2) providing appropriate activities to enrich his experience and challenge his efforts. The methods and procedures for doing these things will vary greatly with the particular situation, and probably very fortunately so; this is what makes teaching a creative art, not a mechanical process. If the group is homogeneously superior, or largely so, that is one thing; if it contains only a few able students, that is another thing; if it is a mixed group, the problem is still different. Methods will also depend upon the type of lessons and subject matter being covered.

Supplementary problems or topics are readily found for practically every unit being studied. Those requiring superior ability could be assigned as a challenge to the better students. Yet they may not be an appropriate challenge for these particular students; nor is the offer of extrinsic reward, such as a higher mark, or extra credit, a sufficient or effective motivation. Gifted students differ more among themselves than do the average students, as to interests, needs, and attitudes. They require more individualized attention and adroit handling. In order to handle them effectively the teacher should know their interests and their ambitions rather well. Supplementary work should be carefully hand-picked for them; topics allied to their hobbies or special interests would be preferable. Assuming a mixed group with a minority of able students, the main motivation efforts should be directed to the entire class first. Assuming the majority

are thus challenged the teacher then, pushing the topic farther, higher, and into the by-ways, reaches ramifications to which only the better students react. This response by the abler students, in the form of spirited discussion, blackboard demonstrations, etc., also incites the slower students to improvement. But, more important, these "extras" are the roots from which talented efforts should spring and serve as the beginning of superior achievements by the abler ones.

The "follow-through" is then most important if the teacher is to capitalize the advantage gained for the abler students. Here a number of things can be done. Either small group, or individual conferences, or both, are effective in suggesting additional research or investigation leading to interesting and satisfying results. These can be scheduled before or after school, or during free periods. In this supplementary assignment work, the judgment, ingenuity, background and training of the teacher are truly tested. It is fascinating work, creative teaching of the highest order, and inwardly rewarding. Here will be started on their careers our future Pascals, Einsteins, or Decarteses.

Such classroom experiences can serve as the genesis of group ex-curricular activities, such as mathematics clubs, assembly programs, and the like. Such programs and organizations serve other useful purposes, such as character building, initiative, self-expression, and sportsmanship, which every school aims to develop. However, if the organization is started by a group interested primarily in mathematics, the programs which are planned by it will more likely follow this useful pattern, and much valuable mathematics will be covered thereby.

The plethora of supplementary material in current books and periodicals today obviates the need here of presenting any exhaustive suggestions. Several professional journals such as *The Mathematics Teacher, School Science and Mathematics, Scripta Mathematica, The Mathematical Monthly,* contain a wide variety of source material as well as bibliographies compiled topically or by subject-matter areas.

Some further remarks concerning the selection and administration by the teacher of this student work might however be in order. A topic assigned to an individual student should be geared to his interest and ability. Once the proper matching has been done between the proper student and the proper reference material, the process of following it up is important. Assistance, encouragement, and further motivation may be needed. Ultimately some form of presentation is planned. This requires organizing and selecting the interesting parts to tell, either before the class, the mathematics club, or other groups. In case it does not possess appeal to a group, then the audience may be confined to the teacher alone. However, it is more profitable if given before a group.

An interesting mathematics club program can be planned on some central theme, if individual students present separate phases of the topic, culminating in some interesting generalization, or its application to some practical use, instrument, or process, in which the student audience can in some way participate. This is more easily accomplished when visual aids are included in the presentation. Topics that have been found popular along this line and using visual aids include:

The Tower of Hanoi, illustrating geometric series, powers of two, and the binary system.
The Chinese Rings, a somewhat more intricate application of the same topics.
Linkages, of many types, illustrating numerous principles in classical and modern geometry.
Construction of conic sections, in a variety of ways, ranging from mechanical devices and paper-folding to projective geometry, and their important uses.
*Applications to art, using mean and extreme ratio, the divine proportion and dynamic symmetry.
Applications to science, engineering and industry.

* The starred topics would require a minimum of visual aid equipment.

Mathematics in music, including the harmonic mean, the equitempered scale based on $\sqrt[12]{2}$ and periodic functions representing simple and composite tones.
Proofs of Pythagorean theorem, using cut-outs and other devices.
Pi, developed historically, geometrically, analytically, by probability theory, infinite series, and most accurately by modern electronic conputers.
*e, the basis of natural growth, illustrated by continuous compounding of interest; its derivation by either algebra or calculus.
Crystallography and the study of regular solids.
Trisection of angle, its history and influence on discovery of conics and higher curves.
Elements of non-Euclidean geometry.
Measuring instruments and devices of all kinds, both primitive and modern.
Computing instruments and devices of all kinds, both primitive and modern.
*Cryptic arithmetic, involving the filling in of missing digits in an arithmetical example.
*Magic squares of different orders and kinds.
*Mathematical games such as Nim, Chinese chess, etc., using the principle of powers of two.
*Logic, or "thought" problems, involving deductive, or indirect, reasoning.
*Mathematical recreations and fascinating problems such as Diophantine cases.

Provision should be made for developing the talents of students gifted in mechanical, manual, or other special abilities. Making of instruments, models, or other teaching aids is helpful to them, the class, and the teacher, and certainly should be encouraged. Students have access to wood, machine, or other shops either at home or in other departments in the school, and other teachers are generally glad to correlate with such projects. Definite plans should originate with the mathematics teacher, however, to assure the desirable outcomes. It is helpful to have an extra room, as a mathematics laboratory or store-room, if the school can provide it, where student projects, and other teaching aids and equipment, can be organized and stored. The displaying of equipment and models in showcases and on bulletin boards is a necessary adjunct to an attractive classroom and is an appropriate task for students with artistic talents.

Assemblies, parent-teacher programs, mathematics plays, fairs, exhibits, and similar activities are valuable for motivation and for popularizing the subject among students not presently enrolled in mathematics classes. Such activities are commonly the outgrowth of special projects performed by able and enthusiastic mathematics students.

As competition is a good incentive for superior achievement, mathematics contests always pay off scholastically. These can be intra-class or on a school-wide basis. Mathematics clubs can sponsor them effectively with proper teacher guidance. They can be organized on various levels to attract beginning as well as advanced students. *Oral* contests are more entertaining for public presentation than are *written* ones. They require careful teacher participation both in judging and administering the contest. A goodly supply of short pointed questions, of various levels of difficulty, but which lend themselves to a neat and elegant attack and solution, are a helpful resource of such a teacher.

Now in closing, I frankly admit that I have said nothing new. As Montaigne, the French philosopher, said, "I have gathered a nosegay of flowers in which there is nothing of my own but the string that ties them." And to prove this, may I close by quoting Plato who, 2200 years ago, said: "Do not then train boys to learning by force and harshness; but direct them to it by what amuses their minds, so that you may be the better able to discover with accuracy the peculiar bent of the genius of each."

"But even mathematics, the most noble and abstract of all sciences, has its crown in the air, but its roots deep in the earth on which we live."

—In a letter to GALOIS from his father.

The Solution of a Radical Equation

By F. S. NOWLAN
University of Illinois, Navy Pier, Chicago 11, Illinois

THE relationship

$$\sqrt{a}\sqrt{b}=\sqrt{ab}$$

fails when both a and b are negative. For example, if $a=-1$, $b=-1$, the formula would give

$$\sqrt{-1}\sqrt{-1}=\sqrt{(-1)(-1)} \quad \text{or} \quad i^2=1,$$

whereas $\qquad i^2=-1.$

It follows that such a relationship as

$$\sqrt{1-x}\sqrt{3-2x}=\sqrt{3-5x+2x^2}$$

is not identically true. Thus, for $x=2$, it would give the incorrect statement

$$\sqrt{-1}\sqrt{-1}=1, \quad \text{namely,} \quad i^2=1.$$

Similar misstatements are found in the solutions of radical equations in standard textbooks. For example, given the equation

$$\sqrt{1-x}+\sqrt{3-2x}=2\sqrt{5-3x},$$

the usual procedure is to square and incorrectly write

$$(1)\qquad (1-x)+(3-2x)+2\sqrt{3-5x+2x^2}=4(5-3x)$$

whence

$$(2)\qquad 2\sqrt{3-5x+2x^2}=16-9x.$$

Then, squaring again, and collecting terms, one obtains

$$(3)\qquad 73x^2-268x+244=0.$$

The equations (1) and (3) are satisfied by $x=2$, but this value fails to satisfy the equation (2). It is a questionable principle that two wrongs make a right.

In a correct solution, one would square the members of equation (1) in the form

$$(1-x)+(3-2x)+2\sqrt{1-x}\sqrt{3-2x}=4(5-3x).$$

Then,

$$(2)'\qquad 2\sqrt{1-x}\sqrt{3-2x}=16-9x.$$

A third squaring, and collecting terms gives

$$(3)\qquad 73x^2-268x+244=0.$$

The equations (1), (2)′, and (3) are satisfied by $x=2$.

HAVE YOU READ?

Murray, Ellsworth Neil, "High School Mathematical Preparation for the United States Naval Academy," *California Journal of Secondary Education,* May 1953, page 265.

The question is continually raised, "Are algebra and geometry being taught in our secondary schools with sufficient effectiveness to meet the needs of the students?" In this article Mr. Murray reports his findings after studying this question in regard to the entrance requirements of the United States Naval Academy. His findings are satisfying and encouraging. His implications are thought-provoking, especially in the area of geometry. To satisfy yourself concerning these implications you may want to study this problem further.—PHILLIP PEAK, Indiana University, Bloomington, Indiana.

The President's Page

EXCERPTS FROM A REPORT OF A COMMITTEE OF THE ILLINOIS SECTION OF THE M.A.A.

References to instances of and the desirability of cooperation between secondary school and college teachers of mathematics have appeared on this Page several times in the past eighteen months. The National Council wishes to encourage these activities in every way possible. As a means of doing this and, more importantly, of bringing to a wider audience some sound recommendations on this problem, this Page is given to excerpts from a Report of a committee of the Illinois Section of the Association. The Report was presented to the Illinois Section May 9, 1953. Members of the Committee which prepared the Report were: Franz E. Hohn, University of Illinois, Chairman; Mary Entsminger, Laboratory School, Southern Illinois University, Carbondale; Martha Hildebrandt, Proviso Township High School, Maywood; Alice Seybold, North Central College, Naperville; and Henry Swain, New Trier Township High School, Winnetka.

While I am not in complete agreement with all of the paragraphs of the Report quoted, especially in regard to degree of the "powerful and dangerous trend" to which the Report refers, I am very happy to give publicity to it as the sincere expression of an important Committee of an important organization. The National Council can look forward to further developments in Illinois, as a result of the Report. Information on acceptance or rejection of the Recommendations is not available to me at the time of preparing this copy, except for recommendation (c) of the Report. I am glad to report that the Mathematical Association of America has been cooperating in the planning of the proposed national high school periodical.

It is to be regretted that the full Report could not be carried on these pages. A limited number of copies of the Report can be obtained from Professor Franz E. Hohn, Department of Mathematics, University of Illinois. Sections of the Report on Problems Relating Primarily to the Teacher, The Problem of Cooperation, and the Activities of the Committee are not included in the following excerpts.

John R. Mayor

Report to The Illinois Section of the Mathematical Association of America of its Committee on the Strengthening of Mathematics Teaching

The Nature of the Problem and the Task of the Committee

The Problem as the Committee Sees It

On the basis of observation by its members and by others, the committee believes that there exists a powerful and dangerous trend against solid content in education, a trend which extends from the elementary schools through the high schools and which involves all subject-matter fields. It is important therefore to encourage teachers of mathematics, and of other subjects as well, to cooperate in a united effort to improve the quality of teaching and learning, and to improve the significance of the subject-matter taught, in the schools.

The Brighter Side of the Picture

The committee recognizes, of course, that many teachers, administrators, school board members, and other citizens are in fact doing splendid work on behalf of good education, frequently in the face of serious difficulties. To these it pledges all the support and cooperation it can give. Indeed, it hopes that its conclusions and recommendations will give voice to many of their problems and many of their aspirations.

The Committee's Task

Apart from a recognition of the seriousness and the scope of the problem, some aspects of which are detailed below, and apart from a recognition of the strength of the forces working against a sound educational program, the committee believes that its function is not primarily to join with the mounting roll of critics, but rather that its obligation is to develop—and, so far as possible, to implement—concrete, workable proposals. In short, the committee believes that its function is to help begin the task of reconstruction.

Problems Relating Primarily to the Student

The Problem of Proper Guidance

Assuming competent instruction, the primary problem in the case of the student is one of proper guidance. We believe that much advice to the effect that mathematics is not necessary or desirable is based on the misconception that the subject is inherently difficult and dull for most students. However, the phenomenal success of recent mathematical TV programs, and the widespread and perennial interest in mathematical puzzles and riddles is proof of a greater latent interest in mathematics than is ordinarily suspected of the general run of mankind. This interest can be kept alive and developed in students only by good teaching and by proper emphasis on the importance of mathematics in human affairs. It is therefore a duty of the mathematics teacher and the administrator properly to inform students, patrons, and those responsible for guidance, of the vital significance of the subject.

Importance of Mathematics in Modern Life

The fact of the matter is that in every area of scientific investigation today, as well as in many types of skilled labor, a certain body of mathematical skills is required for effective participation. Moreover, these requirements are rapidly increasing and will continue to do so as our civilization becomes increasingly technical. Every student and parent should know this, and moreover, every high school student should continue his mathematical training to the full extent of his ability. Whether or not he goes on to college—and this is something he may not be sure of himself—this training will serve him in good stead if it is sound material and is well taught. What the colleges, science, industry, and the vocation of citizenship need and cry for today is the help of thinkers, not dilettantes or robots.

Aids to Proper Guidance

There are various aids available for a proper guidance program. There is a guidance bulletin prepared by the National Council of Teachers of Mathematics, there is the widely circulated bulletin on the mathematical needs of prospective engineering students prepared by the University of Illinois, there are circulars prepared by industry, such as General Electric's *Why Study Mathematics?*, and there are helpful articles in such journals as The Mathematics Teacher and the *American Mathematical Monthly*. A pamphlet listing these aids and suggesting proper methods in mathematics guidance would be of great value if it could be circulated extensively at little cost among educators, patrons, students, and those responsible for guidance.

Guidance Through Informal Lectures

The committee is aware that not all guidance is planned as such, but may

rather be a by-product of other experiences. As an example of this, the committee has had the opportunity to observe how effectively good popular lectures on mathematics can represent its importance in the educational process as well as in modern life and thought. We therefore urge mathematics teachers and interested laymen to utilize every opportunity to speak in behalf of the subject, whether it be to groups of students or to groups of patrons.

The Responsibility of the Teacher

No amount of proper guidance can solve the problem, of course, in the absence of effective and inspirational teaching. Many teachers have found that striking exhibits, properly used, will create interest in and convey the importance of mathematics. There is unlimited room for ingenuity here. Frequently topics commonly presented only in a formal way can be made both meaningful and inspirational by a few simple indications of how one would use these ideas in the factory, the laboratory, the market place, or in the understanding of nature. Ideas for enrichment materials of this kind are to be found in the professional periodicals, with which every teacher should be thoroughly familiar. Every teacher ought also to feel an obligation to share with other teachers good ideas which may not be widely known, through the medium of such periodicals. These observations point up the extent to which the problem reverts to the classroom teacher. Sound material, inspirationally taught, is one of the best means of advertising our subject and represents the greatest contribution to good education we can possibly make.

Problems Relating to Schools and Curricula

Correspondence Courses for High School Students

The committee feels a great concern for the high school student who needs and desires adequate mathematical training but whose school cannot or will not offer the necessary courses. The committee believes that genuine intellectual ability is so precious a quality that it ought never be allowed to suffer for lack of the opportunities necessary for proper development. We believe therefore that the universities should follow the lead of such schools as the University of Wisconsin in developing and advertising energetically a program of correspondence courses in those secondary school subjects which are frequently neglected.

One effect of advertising such a program would of course be such as to make students more fully aware of the importance of mathematics in their preparation. It would also make school boards more fully aware of their responsibility for providing proper curricular opportunities. In the long run, one would therefore expect such a program to be self-liquidating, which is as it should be.

The Problem of a Good Curriculum

It is a matter of observation that guidance, instruction, and administration in certain high schools is of such a high calibre that the graduates of these schools do outstandingly well in employment requiring technical skills as well as in college work and beyond. These are schools in which thorough teaching of fundamentals has been emphasized. This fact alone is evidence that the critics of modern education are dealing with genuine issues. Moreover, it emphasizes the need for making teachers, school officials, and patrons aware of the statistical flaws in the now notorious *Eight-Year Study*, flaws which render its conclusions of little value. The fact of the matter is that as yet nobody has demonstrated that a properly balanced program of subject-matter courses, with the emphasis on methods of creative and critical thinking, is not the best preparation for college work or for life.

Problem of Maintaining High Standards

In view of the above observations the committee wishes to go on record as favor-

ing the subject-matter type of curriculum as opposed to the core type of program, specific subject-matter requirements for high school graduation and for college entrance, and mathematics for all those able to comprehend it. This does not mean that the committee is opposed to changes in content of courses, in methods of instruction, or in methods of evaluating achievement. It does mean that the committee is opposed to such changes as imply lowered standards of scholarship and accomplishment. Certainly educational research ought to enable us to improve these standards rather than to urge us to lower them. When it does the latter, it runs counter to the realities of our times, and is based on false premises, faulty observation, or errors of logic. It is indeed not the peculiar whim of the professors, nor of the college administrators, that standards be kept high; it is the demand of life itself. The real life-adjustment is no retreat into a core of mediocrity, but rather an honest facing of the fact that knowledge and the wisdom to use it constructively are the essence of survival.

Recommendations of the Committee

In order to implement further its conclusions, the committee recommends to the Illinois Section of the Mathematical Association of America the following actions:

(*a*) that the Illinois Section recommend to the Mathematical Association of America that it appoint a Committee on Cooperation with Industry to capitalize on industry's known willingness to invest money in advertising the need for good mathematical and scientific training at the high school level and in workshops for high school teachers, designed to inform them of ways in which mathematics is applied to practical problems

(*b*) that the Illinois Section and the Illinois Council of Teachers of Mathematics explore the possibility of cooperating in the organization in Illinois of an annual summer workshop for high school teachers

(*c*) that the Illinois Section recommend to the MAA that it offer the NCTM aid, support, and cooperation in the proposed launching of a national high school mathematics periodical

(*d*) that the members of the Illinois Section interested in teacher training be urged to offer to prepare for the NCTM, pamphlets on such topics as "How to Organize a Workshop," "How to Make a Good Assignment," and others of the many worthy topics for which the NCTM has been unable to find authors

(*e*) that the Illinois Section urge its members to join the Illinois Council of Teachers of Mathematics and the National Council of Teachers of Mathematics as an expression of support of efforts of these organizations to improve the teaching of secondary mathematics

(*f*) that the Illinois Section recommend to the Mathematics Department of the University of Illinois that it prepare and arrange for the distribution of a companion to the bulletin on the mathematical needs of prospective engineers, the new bulletin being designed to list the mathematical needs of students planning to enter fields of study other than engineering

(*g*) that the Committee for the Strengthening of Mathematics Teaching be made, officially as well as in effect, a joint endeavor of the Illinois Section and the Illinois Council

(*h*) that the Illinois Section consider a moderate increase in its dues in order to be able to defray reasonable expenses of such committees as it may appoint, and finally

(*i*) that the Illinois Section approve this report and authorize its circulation among concerned members of

- the several mathematical organizations
- the public and the press
- the various agencies of the University of Illinois
- the legislature and committees thereof and
- the United States Office of Education.

The National Council Affiliated Groups

By Mary C. Rogers, *Chairman of Committee on Affiliated Groups*
Roosevelt Junior High School, Westfield, New Jersey

Early in the summer of 1952, plans were considered for the reorganization and expansion of the work of the Committee on Affiliated Groups. These plans suggested the distribution of Affiliated Groups into geographical areas or regions with a Regional Representative directly responsible for the leadership of local groups within his specific area in matters pertaining to their relationship with the National Council of Teachers of Mathematics. A General Chairman would act as Coordinator and Advisor, and be responsible to the National Council for the successful functioning of the Affiliated Groups organization. The office of the Chairman would be the "clearing house" of inter-group information. Copies of all reports would be kept on file in this office. The Chairman would have charge of the finances of the Committee on Affiliated Groups.

The fifty-eight Affiliated Groups were to be distributed as equally as possible into four Regional divisions—based on geographical location, relative population of these areas, and the number of Affiliated Groups (existing and potential) in these areas. The following committee personnel were recommended:

Northeastern Area	Jackson B. Adkins
Southeastern Area	William A. Gager
Southwestern Area	Ida May Bernhard
Northwestern Area	Donovan A. Johnson
General Chairman	Mary C. Rogers

These tentative plans were submitted to the Board of Directors of the National Council at their summer meeting in Exeter, New Hampshire, in August, 1952. The plans and personnel were approved by the Board and a two-year term of service assigned to committee members. The Area distribution is as follows:

Northeastern Area	Groups Affiliated
Connecticut, Maine, Massachusetts, New Hampshire, Rhode Island, Vermont	2
Delaware*	1
District of Columbia	2
New Jersey	1
New York	3
Ontario	1
Pennsylvania	3
	13

Southeastern Area	Groups Affiliated
Alabama	1
Florida	4
Georgia	1
Kentucky	1
Louisiana, Mississippi	1
Maryland	2
North Carolina	1
South Carolina	1
Tennessee	1
Virginia	2
West Virginia	1
	16

Southwestern Area	Groups Affiliated
Arizona	1
Arkansas	1
California	1
Colorado	1
Kansas	2
Missouri*	1
Nevada	0
New Mexico	1
Oklahoma	3
Texas	4
Utah	1
Wyoming*	1
	17

* States having groups newly affiliated in 1952–53.

Northwestern Area	Groups Affiliated
Idaho	0
Illinois	4
Indiana*	2
Iowa	1
Michigan*	2
Minnesota	1
Montana	0
Nebraska	1
North Dakota	0
Ohio	3
Oregon	1
South Dakota	1
Washington	0
Wisconsin	1
	17

Early in September, 1952, the General Chairman sent letters of greeting to all Affiliated Groups. These letters expressed sincere appreciation from National Council for the fine work being done by the Local Groups—for the strong support these groups are giving the National Council program of service. The letter also explained the new organization of the Committee on Affiliated Groups, introduced the Regional representatives, and offered whatever assistance the Local Groups desired, if it was at all feasible and possible.

About the first of October, the General Chairman sent to the Regional Representatives materials to be used in their work with their Groups, together with a detailed outline of the year's plans and program of activities. The Representatives established direct contact with their Groups shortly thereafter, sending them Affiliation Renewal forms and information concerning Affiliated Group activities planned for the current year.

The year 1952–53 has been a very successful one for the Affiliated Groups organization. This success has been due in large part to the outstanding interest shown by the leaders of the many Affiliated Groups, and to their enthusiastic support of all activities and services presented to them as a part of the Affiliated Groups program. An eq[illegible] amount of credit goes to the Regional Representatives. The National Council and the General Chairman are most grateful to these leaders for their prompt and efficient co-operation at all times—for the fine services they have rendered the Groups in their respective areas. They are also to be commended for the excellent Newsletters which they have prepared with the assistance of their local groups, for their helpful articles in The Mathematics Teacher, and for the special programs provided by them at all National Council meetings.

Affiliated Groups Newsletters

Nov., 1952 .... William A. Gager and Southeastern Area.
Jan., 1953 .... Jackson B. Adkins and Northeastern Area.
March, 1953 ...... Donovan A. Johnson and Northwestern Area.
May, 1953 .... Ida May Bernhard and Southwestern Area.

Articles in The Mathematics Teacher

Oct., 1952 .... Report of the Third Delegate Assembly, Mary C. Rogers
Dec., 1952 .... Officers of the NCTM Affiliated Groups, Mary C. Rogers
Jan., 1953 .... The Fourth Delegate Assembly, Mary C. Rogers
Feb., 1953 .... Affiliated Groups Activities, Donovan A. Johnson
March, 1953 ...... The National Council Affiliated Groups, William A. Gager
April, 1953 .... The Ford Experiment, Ida May Bernhard and Mary Lee Foster
May, 1953 .... Affiliated Group Activities, Jackson B. Adkins

Affiliated Groups Sections—NCTM Meetings

Lincoln, Nebraska—December 30, 1952

Presiding: Mary C. Rogers, Roosevelt Junior High School, Westfield, New Jersey

"*A Technique for Introducing Decimals*," Alice Rose Carr, Ball State Teachers College, Muncie, Indiana

"*Learning Devices for the Junior High School Student*," Veryl Schult, Wilson Teachers College, Washington, D.C.

"*Building an Understanding of Number Concepts*," Donovan A. Johnson, Uni-

versity High School, University of Minnesota

"*My Six Teaching Aids for Plane Geometry,*" Amelia Richardson, McKeesport, Pennsylvania

"*Some Practical Applications of Conic Sections,*" W. V. Unruh, Mission High School, Merriam, Kansas

Atlantic City, New Jersey—April 11, 1953

Panel Discussion: "*Meeting the Needs of the Gifted Child*"

Donovan A. Johnson, University of Minnesota, Chairman

Myrl H. Ahrendt, NCTM Executive Secretary, Washington

Mamie Auerbach, John Marshall High School, Richmond, Va.

Kenneth E. Brown, Specialist for Mathematics, U.S. Office of Education

John Schacht, Bexley High School, Columbus, Ohio

Louis Thiele, Divisional Director, Exact Sciences, Detroit

Kalamazoo, Michigan—August 24, 1953

Topic: "*Articulation of Mathematics with Business and Industry*"

Presiding: Mary C. Rogers, Roosevelt Junior High School, Westfield, New Jersey

"*The Committee on Cooperation of Mathematics Education with Industry—a Progress Report,*" Phillip S. Jones, University of Michigan, Ann Arbor, Michigan

"*Mathematics in Industrial Research,*" Arvid Roach, Research Laboratories, General Motors Corporation, Detroit, Michigan

"*Mathematics for Industry at the Technical Level,*" W. D. Merrifield, Director of Industrial Education, Chrysler Corporation, Detroit, Michigan

Groups Affiliated Since May, 1952

Delaware Council of Mathematics Teachers

Wyoming Council of Mathematics Teachers

Missouri Affiliated Groups of NCTM

Gary Council of Teachers of Mathematics

Michigan Council of Teachers of Mathematics

The Association of Mathematics Teachers of New York State will complete affiliation by September 1, 1953. We are in correspondence with five other Groups who are planning to join us in the near future.

A great deal of interesting and helpful information concerning activities has been reported to the Affiliated Groups Committee during the past year. The Renewal Reports, local Bulletins and Newsletters, local meeting programs, and outlines of activities being sponsored have all been read with pleasure and with an appreciation of the excellent services being rendered by local groups to the teachers and students of mathematics in their localities.

The Regional Representatives have summarized and edited much of this information for publication in the Affiliated

Area and Representative	Affiliation Renewals	New Affiliations	Delegates at Delegate Assembly
1. Northeastern Jackson B. Adkins	12	1	12
2. Southwestern Ida May Bernhard	15		15
3. Southeastern William A. Gager	16	0	13
4. Northwestern Donovan A. Johnson	14	2	12
	57	5	52

Groups Newsletters. These Newsletters are sent regularly to each of the 320 officers of the 63 Affiliated Groups, thus making possible an exchange of ideas and experiences that is nationwide. Copies of these publications are also sent to National Council Officers and Board members and to the Executive Secretary at the Washington Office.

The publication and dissemination of these Newsletters is in itself a sizable task. An Affiliated Groups Editor has been named to carry on this work. He is H. Glen Ayre, Western Illinois State College, Macomb, Illinois.

Interesting data concerning our Affiliated Groups includes the following:

State Groups	34
Regional Groups	5
County Groups	6
City Groups	18
	63

Groups reporting a local membership	
less than 50	13
between 50–75	16
between 76–100	4
between 101–200	12
between 201–300	6
between 301–400	2
more than 400	3
not reporting membership	7
	63

Average number of meetings per year, per organization	3
Organizations meeting in one session per year	12
Organizations meeting once a year, but in 2–3 day conventions	3
Organizations sponsoring district meetings in addition to regular State meetings	7
Organizations reporting the sponsorship of local publications	13

Official Delegates
Fourth Delegate Assembly
Atlantic City, New Jersey

Mary C. Rogers served as chairman of the Fourth Delegate Assembly. She was assisted by the Regional Representatives, Jackson B. Adkins, Ida May Bernhard, William A. Gager, and Donovan A. Johnson. Jeannette Garrett and Dorothy Sward were secretaries of the Assembly. Special reports were presented by M. H. Ahrendt, Executive Secretary, Washington, D.C.; Kenneth E. Brown, Specialist for Mathematics, U.S. Office of Education; E. H. C. Hildebrandt, Editor of The Mathematics Teacher; F. L. Wren, Chairman of the Yearbook Planning Committee; John R. Clark, Editor of Twenty-second Yearbook; Henry W. Syer, Chairman of Committee on Publications of Current Interest; Agnes Herbert, Chairman of Committee on Cooperation with N.E.A.; George E. Hawkins, A.A.A.S. Cooperative Committee on Science and Mathematics; Phillip S. Jones, Chairman of Committee on Cooperation of Mathematics with Industry; Henry Van Engen, Chairman of Research Committee; Hubert B. Risinger, Chairman of Speakers' Bureau; Madeline D. Messner, Chairman of Traveling Exhibit.

The official delegates were:

Mathematics Department, Alabama Education Association
  Jeannette Garrett, Birmingham, Alabama
Arizona Mathematics Association
  Robert S. Fouch, Tempe, Arizona
Arkansas Council of Teachers of Mathematics
  Helen Graham, Fayetteville, Arkansas
California Mathematics Council
  Maude Coburn, Oakland, California
Colorado Council of Teachers of Mathematics
  H. W. Charlesworth, Denver, Colorado
Delaware Council of Mathematics Teachers
  Ruth Lee Green, Wilmington, Delaware
Florida Council of Teachers of Mathematics
  Kenneth P. Kidd, Gainesville, Florida
Dade County Council of Teachers of Mathematics
  Florence J. Shaffer, North Miami, Florida
Pinellas County Council of Teachers of Mathematics
  Annie Mae Hendry, St. Petersburg, Florida
Hillsborough County Mathematics Council
  Mrs. Lorraine Sewell, Tampa, Florida
Georgia Panel, National Council of Teachers of Mathematics
  Isabel Kinnett, Macon, Georgia
Illinois Council of Teachers of Mathematics
  Henry Swain, Winnetka, Illinois
Men's Mathematics Club of Chicago and Vicinity
  Lee Dulger, Harvey, Illinois
Women's Mathematics Club of Chicago and Vicinity
  Geraldine Kauffman, East Chicago, Indiana
Gary Council of Teachers of Mathematics
  Wilma S. Flewelling, Gary, Indiana
Indiana Council of Teachers of Mathematics
  Mrs. Mildred Saltzman, Newburgh, Indiana
Iowa Association of Mathematics Teachers
  Paul Shaw, Waterloo, Iowa
Kansas Association of Teachers of Mathematics
  Gilbert Ulmer, Lawrence, Kansas
Wichita Mathematics Association
  Lucy E. Hall, Wichita, Kansas
Mathematics Section of the Maryland State Teachers Association
  Herbert R. Smith, Baltimore, Maryland
Prince George's County Teachers Association Mathematics Section

William Lynn, Hyattsville, Maryland
Detroit Mathematics Club
Lucille Martin, Detroit, Michigan
Michigan Council of Teachers of Mathematics
Geraldine Dolan, Detroit, Michigan
Minnesota Council of Teachers of Mathematics
Donovan A. Johnson, Minneapolis, Minnesota
Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics
Mary Ann Turner, Greenwood, Mississippi
National Council of Teachers of Mathematics Nebraska Section
Edith Ellis, Lincoln, Nebraska
Association of Teachers of Mathematics in New England
Barbara Betts, Swampscott, Massachusetts
Association of Mathematics Teachers of New Jersey
Carl N. Shuster, Trenton, New Jersey
Nassau County Mathematics Teachers Association
Alfred T. Anderson, Garden City, New York
Association of Teachers of Mathematics of New York City
Barnett Rich, New York, New York
Suffolk County Mathematics Teachers Association
Mrs. Harriet Burgie, Sayville, New York
Department of Mathematics, North Carolina Education Association
Annie John Williams, Durham, North Carolina
Mathematics Club of Greater Cincinnati
Eleanor Graham, Cincinnati, Ohio
Ohio Council of Teachers of Mathematics
Oscar Schaaf, Columbus, Ohio
Oklahoma Council of Teachers of Mathematics
James H. Zant, Stillwater, Oklahoma
Mathematics Council of Oklahoma City Teachers
James H. Zant, Stillwater, Oklahoma
Tulsa Council of Teachers of Mathematics
James H. Zant, Stillwater, Oklahoma
Ontario Association of Teachers of Mathematics and Physics
James Kerr, Toronto, Ontario
Association of Teachers of Mathematics of Philadelphia and Vicinity
M. Albert Linton, Jr., Philadelphia, Pennsylvania
Mathematics Teachers Association of Western Pennsylvania
Helen B. Malter, Pittsburgh, Pennsylvania
Pennsylvania Council of Teachers of Mathematics
Catherine A. V. Lyons, Pittsburgh, Pennsylvania
Mathematics Section, East Tennessee Education Association
F. L. Wren, Nashville, Tennessee
Greater Dallas Mathematics Association
Patricia Copley, Dallas, Texas
Beaumont Mathematics Association
Joyce Benbrook, Houston, Texas
Houston Council of Teachers of Mathematics
Edward B. Adams, Houston, Texas
Texas Council of Teachers of Mathematics
Joyce Benbrook, Houston, Texas
Mathematics Section of the Virginia Education Association
Louise Matney, Grundy, Virginia
Richmond Section of the National Council of Teachers of Mathematics
Helen M. Hulcher, Richmond, Virginia
Benjamin Banneker Mathematics Club
Mrs. Emma Mitchell Lewis, Washington, D.C.
District of Columbia Teachers of Mathematics
Carol V. McCamman, Washington, D.C.
West Virginia Council of Mathematics Teachers
Mrs. Muriel G. Curry, Glenville, West Virginia
Wisconsin Mathematics Council
John A. Brown, Madison, Wisconsin

Delegate Assembly Alternates

Pinellas County Council of Teachers of Mathematics
Mrs. Bernice Hotmire, Clearwater, Florida
Florida Council of Teachers of Mathematics
Howard L. Gallant, Tampa, Florida
Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics
Houston T. Karnes, Baton Rouge, Louisiana
Mathematics Section of the Maryland State Teachers Association
Janet V. Coffman, Catonsville, Maryland
Minnesota Council of Teachers of Mathematics
Edith Woolsey, Minneapolis, Minnesota
Pennsylvania Council of Teachers of Mathematics
Mabel Love Baker, Penn Township, Pennsylvania
Mathematics Teachers Association of Western Pennsylvania
Chester M. Jelbert, Pittsburgh, Pennsylvania
District of Columbia Teachers of Mathematics
Russell B. Coover, Chevy Chase, Maryland
Department of Mathematics, North Carolina Education Association
W. W. Rankin, Durham, North Carolina

HISTORICALLY SPEAKING, - -

Edited by Phillip S. Jones
University of Michigan, Ann Arbor, Michigan

Bibliographia Historica—I.

A bibliography of the history of mathematics would be a tremendous task involving too many languages and too many too technical works to merit space here. However, a bibliography of recent, readable historical materials available and usable by students in secondary schools and undergraduate colleges would not be so long and would be quite pertinent here. We hereby solicit such materials and, for a beginning, we include a list of our own.

The Scientific American, a venerable publication, was purchased in 1947 by a group headed by Gerard Piel formerly of *Life*. Both its purposes and format were revised. Its new policy has led to the printing of a number of articles of mathematical interest. These articles are followed at the back of each issue by a short list of references for further reading, and, in later issues, by letters from readers. In some instances even the book reviews have been short articles of interest in themselves.

The first issue of this new *Scientific American* appeared in May, 1948 as Number 5 of Volume 178. The following is a list of materials of a mathematical-historical nature appearing from May, 1948, through the issue of July, 1953.

Boyer, Carl B., "The Invention of Analytic Geometry." Vol. 180 (January 1949), pp. 40–45.

Cohen, I. Bernard, "Galileo." Vol. 181 (August 1949), pp. 40–47. Not much explicitly about mathematics, but a chapter in the development of theories of the universe, which story as a whole has many points of contact with mathematics.

———, "Books." Vol. 179 (October 1948), pp. 54–59. Reviews of books on the history of science with pictures of Norse runic numerals and diagrams for an ancient system of the universe.

Euler, Leonhard, "The Koenigsberg Bridges." Vol. 189 (July 1953), pp. 66–70. A translation with an introduction by James R. Newman.

Herwitz, Paul S., "The Theory of Numbers." Vol. 185 (July 1951), pp. 52–55.

Morrison, Philip and Emily, "The Strange Life of Babbage." Vol. 186 (April 1952), pp. 66–73.

Newman, James R., "William Kingdon Clifford." Vol. 188 (February 1953), pp. 78–84.

———, "Srinivasa Ramanujan." Vol. 178 (June 1948), pp. 54–57.

———, "The Rhind Papyrus." Vol. 187 (August 1952), pp. 24–27.

———, Reviews of *Makers of Mathematics,* by Alfred Hooper and of *Our Great Heritage,* edited by William Schaaf. Vol. 179 (November 1948), pp. 56–59. Pictures of the title-page of Billingsley's first English language edition of Euclid and of a part of the Rhind papyrus are included. It is in part an essay on "What Is Mathematics?"

———, Review of *Newton's Tercentenary Celebration.* Vol. 179 (July 1948), pp. 57–59, with pictures and a bibliography.

———, "Mathematical Creation." Vol. 179 (August 1948), pp. 54–57. This is really a summary of much of the writing of the famous French mathematician Henri Poincaré on the topic of how one gets a new mathematical idea. This is not strictly history of mathematics, but

is related to one of the most interesting problems of the history of mathematics, "How does a mathematical idea grow? Where do new ideas come from?"

Reid, Constance, "Perfect Numbers." Vol. 188 (March 1953), pp. 84–86.

Santillana, George de, "Greek Astronomy." Vol. 180 (April 1949), p. 44. Both ideas of the nature of the universe and many of the persons mentioned here (Anaximander, Pythagoreans, Eudoxus, Ptolemy, Copernicus, Aristarchus) are closely related to mathematics.

Struik, Dirk J., "Stone Age Mathematics." Vol. 179 (December 1948), pp. 44–49.

Whittaker, Sir Edmund, "Mathematics." Vol. 183 (September 1950), pp. 40–42. This note on modern developments in mathematics is one of a set of articles on various sciences under the general heading of a lead article "The Age of Science: 1900–1950," by J. R. Oppenheimer.

The Oldest American Slide Rule

In section M-4 of the Henry Ford Museum, Dearborn, Michigan, an example of *Palmer's Computing Scale* has recently been put on display. This is a circular slide rule. It contains an eight-inch circular log scale which revolves within a similar circular scale printed on a piece of paper-covered cardboard nearly twelve inches square.

Further investigation shows that the first version of this slide rule was manufactured in 1843, a second and third, in 1845 and 1847. Where the first contained "Directions for Using this Scale" on the back, the second and third contained "Fuller's Time Telegraph," a scale for determining the number of days between dates.

Karpinski in his 1940 bibliography listed only one copy of each, all at Harvard.[1]

Cajori gives the first two dates as 1844 and 1846, commenting that John E. Fuller owned the copyright for the second version.[2] Karpinski agrees that the third version, listed as "Palmer's Computing Scale, improved by Fuller," was copyrighted by Fuller, but also carried Palmer's copyright notice. A number of later versions was published by Fuller under his own name with such varying titles as *Fuller's Computing Telegraph* (1852), *Fuller's Time Telegraph, Fuller's Telegraphic Computer*. A copy of the first of these in the New York Public Library even refers to a London edition.[3] Fuller also published a manual for this rule under varied titles in 1845, 1846 (two editions), and 1848.[4] This probably accounts for the confusion in dates noted above.

Aaron Palmer, however, had published essentially the same book in 1842, before the rule itself appeared, under the title, *A Key to the Endless Self-Computing Scale, showing its application to the different rules of arithmetic, etc.* The first edition was published at Rochester, New York, and a second at Boston in 1844. The copy of the latter in the University of Michigan library promised delivery to subscribers in a few weeks of *Palmer's Endless Self-Computing Scale* which would be published in three styles: for common business calculations, $2; for the higher branches of mathematics, $3; for nautical and astronomical calculations, $5. It also gave four "advantages" for the instrument: "1st. *a complete saving of mental labor;* . . . the most intricate calculations are but a pleasurable exercise of the mind. 2d. *A great saving of time.* Computations requiring from three to four days, are wrought out by this Scale in the incredible short space of one minute [sic!]. 3d. *Complete accuracy* . . . infallible . . . except

[1] L. C. Karpinski, *Bibliography of Mathematical Works Printed in America Through 1850.* (Ann Arbor: University of Michigan Press, 1940), pp. 450, 452.

[2] Florian Cajori, *A History of the Logarithmic Slide Rule and Allied Instruments.* (New York: The Engineering News Publishing Co., 1909), p. 61.

[3] Cajori, *op. cit.*, ix.

[4] Karpinski, *op. cit.*, 471, 472.

through sheer carelessness. 4th. *Mental improvement.* By this Scale, a knowledge of the philosophy of numbers, and their relation to each other, is soon obtained. . . . "

The second American slide rule was contained in *Palmer's Pocket Scale, with rules for its use in solving arithmetical and geometrical problems,* a 48-page pocket size book which appeared in Boston in 1844. The writer has not seen this book or the "7 cm. circular slide rule of brass or cardboard" which Karpinski says accompanied it,[5] but the three 1845 copies (printed in Rochester, New York, Warren, Ohio, and Boston) seen by the writer have a 7 cm. circular disc mounted on the inside of the back cover within a fixed circular log scale which is printed on a cardboard sheet pasted inside the same cover. The latter is $3\frac{1}{2}\times 6$ inches. (See Fig. 1.)

This pocket scale has special gauge points for: the area and circumference of a circle, beer and wine gallons, simple and

[5] *Ibid.*, p. 461.

Fig. 1
(from University of Michigan, History of Science collection)

compound interest for a variety of rates and times, acres, square timber, square yards, square and circle equal in area, inscribed square, cube, triangle, pentagon, hexagon, heptagon, octagon, nonagon, dodecagon, three and four circles inscribed in a given circle.

The book inside of which this scale is fastened is essentially the same as the *Key* mentioned above, differing only by a few corrections and rephrasings. All these books contain an interesting series of recommendations before the preface. Among those recommending it are Benjamin Peirce, a famous Harvard Mathematics Professor, Frederick Emerson, whose *North American Arithmetic* was popular, an engineer, a notary public, a lawyer, and teachers.

In conclusion, the place of this scale and its related books in history may be seen better if we note that the slide rule had been mentioned and sketchily discussed in several arithmetic texts used in the United States, such as George Fisher's *The American Instructor: or, Young Man's Best Companion*, printed by B. Franklin and D. Hall in 1748,[6] as well as in Bowditch's *The New American Practical Navigator* (1802) and Hawney's *Complete Measurer* (1818).[7] The first work printed in America devoted solely to this topic seems to have been George Curtis' *A Treatise on Gunter's Scale, and the Sliding Rule* . . . published in Whitehall, New York, 1824.

All of this reflects something of the interests and needs of a new country, pioneering and trading, as compared with England where John Napier had first published logarithms in 1614, Edmund Gunter discussed his scale in 1620, and William Oughtred published treatments of the circular and straight slide rules in 1632 and 1633 respectively, having, however, been anticipated in the first publication if not in the actual invention of the first circular rule, by Richard Delamain's *Grammelogia* in 1630.[8]

[6] *Ibid.*, pp. 59, 65.
[7] Cajori, *op. cit.*, p. 59.
[8] Florian Cajori, *The History of Mathematics* (New York: Macmillan, 1919), p. 158.

NEWS NOTES

Engineering Booklet

An increasing interest in engineering shown by high school students in recent months has led to the publication of a new booklet about the profession, released recently by Stevens Institute of Technology, Hoboken, New Jersey. Designed to answer questions about the work engineers do, the 16-page publication also discusses the scholastic attainments and special aptitudes which indicate whether a student should seek admission to an engineering college.

In explaining the purpose of the booklet called "What's Engineering?" Dr. Jess H. Davis, president of the Institute, revealed that Stevens and other engineering colleges were getting more requests from high school students for information and for interviews relative to admission than had been received in several years. He said that high school counselors also were being called on to answer more queries about engineering careers. The new booklet was written with an eye to aiding both students and counselors.

Young men are cautioned against going into engineering, despite the promise it seems to hold, unless they show ability in high school mathematics and science. Regarding aptitudes, they are told that a boy who happens to be handy with tools is not necessarily cut out for an engineering career. In order to become an engineer, he must "have the capacity to learn how to design or even improve the radio or motor he enjoys repairing."

DEVICES FOR A MATHEMATICS LABORATORY

Edited by Emil J. Berger
Monroe High School, St. Paul, Minnesota

Anyone who has a learning aid which he would like to share with fellow teachers is invited to send this department a description and drawing for publication. If that seems too time-consuming, simply pack up the device and mail it. We will be glad to originate the necessary drawings and write an appropriate description. All devices submitted will be returned as soon as possible. Send all communications to Emil J. Berger, Monroe High School, St. Paul, Minnesota.

Another Makit Toy Model

Some time ago Miss Nona Mary Allard contributed an article to this section entitled "Devices Made From Makit Toy."* The reader may recall that she described Makit Toy as a "colorful, sturdy, building set manufactured by the W. R. Benjamin Company of Granite City, Illinois." She concluded her article with the suggestion that "the possibilities for constructing working models of Makit Toy are limited only by time and imagination." Pictured in this article is a student from one of the department editor's classes who followed through on this suggestion (Fig. 1).

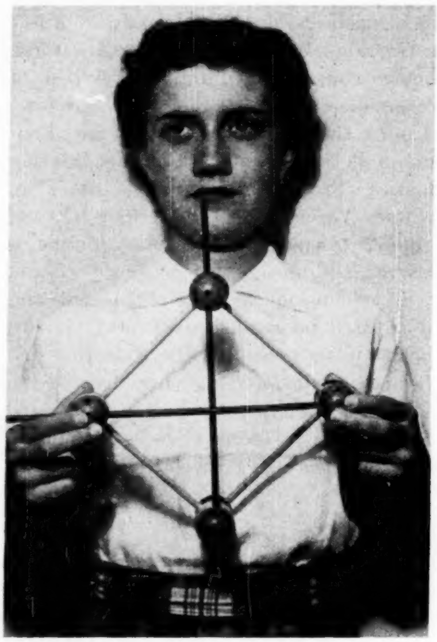

Fig. 1

The model which she is holding can be used to illustrate a number of facts from plane geometry, the most unusual of which is that the diagonals of a rhombus are mutually perpendicular. No matter how the device is extended or collapsed, the diagonals always remain perpendicular to each other.

Building the device from the pieces which are included in a set of Makit Toy is really quite simple, but unfortunately it is rather difficult to describe the procedure in full detail without a live demonstration. However, it is hoped that the following short description of the major features of the device will provide the reader with enough information to proceed with the task of assembling a workable model by himself.

The device consists essentially of an extensible rhombus and two diagonal elements (Fig. 2). The rhombus itself consists of four vertices (made by assembling two wheels), and four plain rods each $5\frac{1}{4}''$ long. The $5\frac{1}{4}''$ length is a standard size for the set. A vertex may be constructed by inserting a short red rod (about $1\frac{3}{8}''$ long) capped with a metal bushing into the center of one wheel, sliding a second wheel over the red rod and finally pushing a col-

* Nona Mary Allard, "Models Made From Makit Toy," The Mathematics Teacher, XLIV (April, 1951), 246–47.

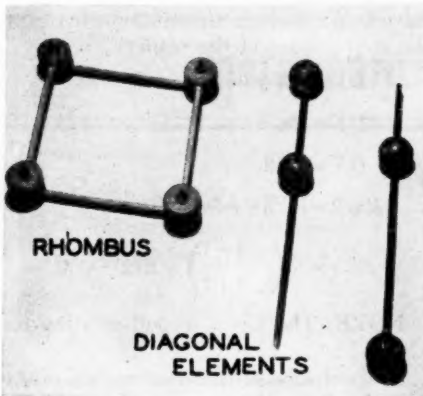

FIG. 2

lar down over the exposed end of the red rod. The exposed end of the short red rod will protrude about $\frac{1}{4}''$ above the second wheel after the vertex is assembled; this information is used again later. The reader will have no difficulty identifying the various parts named above once he has a set of Makit Toy in his possession.

To complete the rhombus so that it can be extended or collapsed as desired, simply observe the direction that the plain rods which meet at a vertex must not be inserted in the edge of the same wheel; i.e., if a vertex is made of one blue and one yellow wheel, insert one rod in the edge of the blue wheel and the other in the yellow.

The two diagonal elements are identical in construction. Included with the set are a number of orange colored balls each of which has 16 surface holes and exactly one "through" hole. Slide a long red rod ($11\frac{1}{4}''$) through the "through" hole of one ball, cap one end of the red rod with a metal bushing, push it into a surface hole of a second ball, and the construction of one diagonal is complete.

Finally place the balls on top of the vertices as shown in Figure 1. As was indicated earlier, the ends of the short red rods that are part of the vertices protrude just far enough ($\frac{1}{4}''$) so that they can be used as anchors for the holes of the orange colored balls. Note that one long red rod crosses over the other.

To manipulate the device it must be picked up and held as shown in Figure 1; this is important. It must be held at the vertices of the diagonal that crosses over on top. The thumb of either hand must be below the vertex wheels and the two forefingers must press the ball down against the vertex wheels. Now by moving the hands together and (or) apart the collapsible and extensible nature of the device may be illustrated at once. In this way it may be shown that no mat'er how the shape of the rhombus is varied the diagonals always remain perpendicular to each other and bisect each other.

E. J. B.

A Problem from Solid Geometry

An innocent appearing problem with which students of solid geometry often encounter considerable difficulty is one which may be stated as follows:

> Given that a sphere with radius R is tangent to the three faces of a trirectangular trihedral angle, what is the radius r of a second sphere that is tangent to the given sphere and also to the three faces of the trihedral angle?

The main source of students' difficulty with this problem seems to be that they are unable to visualize all that is involved; i.e., they are unable to deduce what is implied by that which is given. As a result the drawings which they make often do not include the information needed to solve the problem. The model presented here has proved to be of considerable assistance in helping students "see through" this particular problem. In fact on several occasions the model provided precisely the help needed to enable students to draw a figure which eventually made it possible for them to solve the problem.

The crux of the problem, of course, is that one must consider the plane determined by edge OS of the trihedral angle and the line OT which includes the vertex O of the trihedral angle and the centers X and Y of the small and large sphere respectively (Fig. 3). This plane also includes the line OW (not visible in Fig. 3) which

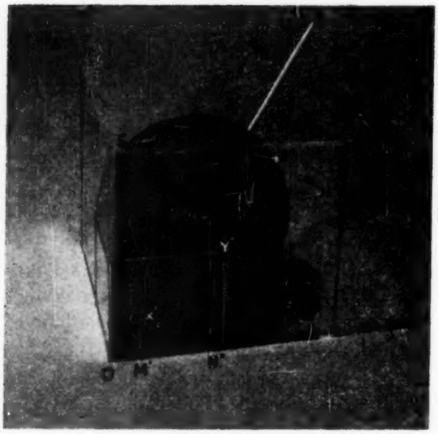

FIG. 3

is the bisector of the "bottom" face angle. Figure 4 indicates the approximate positions of lines OS, OT, and OW. With the aid of the model students will readily see that if X and Y (on OT) are taken as the centers of the two spheres, then M and N are respectively the points of tangency of the small and large sphere on the base plane. So, in terms of Figure 4 the original

FIG. 4

problem becomes that of finding XM $(=r)$ when $YN(=R)$ is given. Further reference to the model should enable the student to see that OY is simply the diagonal of a cube with edge $Y'N'(=YN=R)$ and similarly that OX is the diagonal of a cube with edge $X'M'(=XM=r)$.

Once students discover the foregoing relations their difficulties are usually resolved. We include the solution here for the convenience of the reader:

$$OY=OX+r+R$$

$$OY=R\sqrt{3}$$

$$OX=r\sqrt{3}$$

$$R\sqrt{3}=r\sqrt{3}+r+R$$

$$r=R\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)=R(2-\sqrt{3}).$$

NOTE: There is a second solution for $r>R$.

A description of the construction of the model has been delayed until this point because it was felt that an acquaintance with the use of the model might make such a description more easily understood.

The model as illustrated in Figure 3 was the result of an overnight assignment following a recitation period that pointed up the difficulty already referred to—i.e., inability to visualize all the facts involved. A few of the students (who incidentally were not the best ones in class) had already solved the problem. Of these, one boy agreed to build a model for use by the class on the following day. He produced the one which appears in Figure 3. To construct the trihedral angle he selected three pieces of plastic each $\frac{1}{8}''\times4\frac{1}{2}''\times4\frac{1}{2}''$. For the large sphere he used a rubber ball about $3\frac{3}{8}''$ in diameter, and for the smaller one a wooden ball about $1''$ in diameter. The line OT is a double pointed plastic knitting needle size U.S. Number 3. As indicated in the figure the needle passes through both balls. A hole had to be drilled along a diameter of the $1''$ ball to complete this detail.

The balls are held in position by a piece of scotch-type tape fastened across the open end of the trihedral angle. All white lines which appear on the plastic faces are narrow strips of painters masking tape cut about $\frac{1}{8}''$ wide. This is the entire device—simple in construction, perhaps somewhat crude, but nevertheless quite useful for its particular purposes.

E. J. B.

MATHEMATICAL MISCELLANEA

Edited by

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I Think I Can
I Think I Can

We do not know how many of you recognize "The Little Engine That Could," but that's what your editors feel like. If this train of contributions starts rolling in, we may be over the hilltop and free-wheeling. One of the things that we had not thought of was writing something in July for publication in October. And Clifford was never famed for his interest in lesson plans. However, here are a few items. The major one on Swale's construction just proves how smart one editor (Clifford), who does this part, was to get a co-editor (Struyk) who does the hard work.

88. Swale's Construction

The interesting note by C. N. Mills [*Miscellanea 83*, THE MATHEMATICS TEACHER, XLVI (May, 1953), pp. 344–45] seems to warrant further consideration of Swale's construction. An elementary discussion is outlined below.

First, however, a small correction to *Miscellanea 83*—instead of *Harper's Euclid*, the citation should read *The Harpur Euclid*, by Langley and Phillips (Longmans, Green, and Co., London, 1906).

Swale's construction is a solution of the problem "Find the unknown radius and center of a given circle." The procedure (following loosely the statement on page 199 of *The Harpur Euclid*) is diagrammed in Figure 1. With any point O on the given circle as center, and a suitable radius, draw an arc PQR. With the same radius,

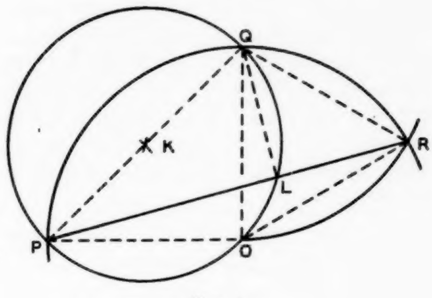

FIG. 1

and with Q as center, draw arc OR, cutting arc PQR at R. Draw line PR, cutting the given circle at L. Then LR is equal to the radius of the given circle. The center can be found by means of two intersecting arcs.

The note in *The Harpur Euclid* ends with the statement, "The proof depends upon III 20." Now Euclid's III 20 is the proposition, "The angle at the center of a circle is double of the angle at the circumference standing on the same arc." In to-day's terminology, a central angle is double an inscribed angle which intercepts the same arc. Applying this fact, $\angle QOR = 2(\angle QPR)$, since O is the center of circle PQR. The equilateral triangle ROQ makes $\angle QOR = 60°$, so that $\angle QPR = 30°$. Thus arc $QL = 60°$, and chord LQ = radius of given circle. (The "problem" as such is solved right here.) By III 20 again, $\angle POQ = 2(\angle PRQ)$. In the given circle $\angle PLQ = \angle POQ$. Hence $\angle PLQ = 2(\angle LRQ)$. But from triangle LQR, exterior angle $\angle PLQ = \angle LQR + \angle LRQ$, so that $\angle LQR = \angle LRQ$. Therefore $LR = LQ$ = radius sought.

This completes the proof of Swale's construction. The following material is an extension of the basic ideas.

We observe that only to achieve the property $LR=LQ$ is it necessary to locate center O of the auxiliary circle on the given circle. If obtaining $LQ=$radius is considered a satisfactory solution of the problem then O may be any point other than K, the lost center. The general aspect of the matter is shown in Figure 2. On the auxiliary circle an arc AB having a specified measure (say $m°$) determines by means of $\angle APB$ an arc XY of $m°$ on the given circle. So the point Q, also, is not indispensable in the construction, but its use reduces by one the number of lines to be drawn.

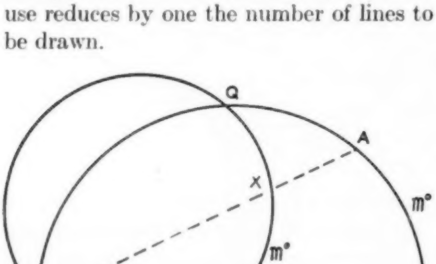

FIG. 2

With point O on the given circle the property $LR=LQ$ is independent of the size of arc QR, but holds for any position of R on the auxiliary circle. (The figure changes considerably as R moves, and details of proof vary, but the essential features of the above proof apply in all configurations—triangle LQR has at vertex R an interior angle equal to half of $\angle POQ$, and at vertex L an exterior angle equal to $\angle POQ$.) Hence the diameter of the given circle is obtained as easily as the radius. (See Fig. 3.) Draw the arc PQ and extend it far enough so that line QO when drawn will cut it at R. Draw PR. This cuts the given circle at L, and makes $LR=LQ=$diameter of given circle.

Now look at Figure 4, where again a point O on a given circle (center K) is the

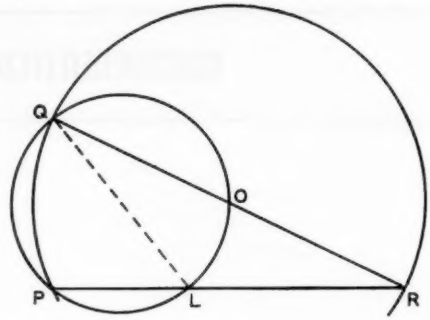

FIG. 3

center of a second (auxiliary) circle. The circles intersect at P and Q. Let $\angle POQ$ measure $m°$. Then the intercepted arc PQ on the given circle measures $2m°$, and the intercepted arc on the auxiliary circle measures $m°$. Let the diameter PS of circle (K) be extended to meet circle (O) at R. Then arcs QR and QS have the same numerical measure, say $x°$.
On (K)

$$2m+x=180.$$

Hence

$$m+x=180-m.$$

To interpret this equation on (O) draw diameter POT in circle (O). Then

$$\text{arc } PR=m°+x°,$$

and

$$\text{arc } QT=180°-m°.$$

Hence

$$\text{arc } PR=\text{arc } QT.$$

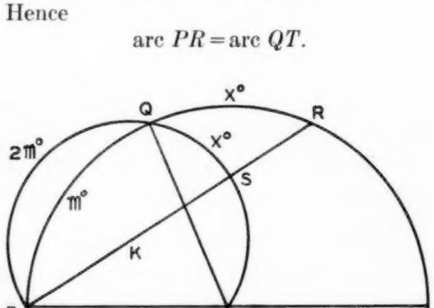

FIG. 4

(Continued on page 524)

WHAT IS GOING ON IN YOUR SCHOOL?

Edited by

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THE METHOD OF SELECTION OF MATHEMATICS TEXTBOOKS IN NEWARK, NEW JERSEY

THE SELECTION of textbooks is of vital importance in any school system. If textbooks are to be accepted by the teaching staff, they must be chosen in a democratic and scientific manner. If textbooks are to be kept up to date, there should be a periodic comparison of new texts with those already in use. These conditions are met by the program of textbook evaluation in the Newark school system.

Each September, the Textbook Council, composed of representative principals and administrators from the central office, selects a mathematics teacher to act as chairman of the "Evaluation Committee for Mathematics" for the succeeding year. Each year one third of the committee of eighteen is appointed for a term of three years. In this way continuity is achieved.

At the first meeting of the 1952 committee, five subcommittees were set up: (1) Junior-High Mathematics, (2) Plane Geometry, (3) Intermediate Algebra, (4) Senior Mathematics, (5) General Mathematics III and IV. The members who were teaching in Junior-High Schools formed the first subcommittee, while each of the other subcommittees was composed, as far as possible, of members who represented the various high schools in the city. The textbooks to be evaluated, recommended by any teacher or publisher, were sent to the committee chairman who distributed them among the chairmen of the appropriate subcommittees for evaluation by their groups.

At a second meeting, the evaluations were turned in *without comment*, and each subcommittee exchanged books with *one* other subcommittee. At a third meeting, the combined reports of all subcommittees were read and discussed. Any additions or deletions were determined finally by the entire committee, and a summary sent to the Textbook Co-ordinator.

The report of the mathematics group, along with the reports of other committees, will be considered by the Textbook Council, and a final list of approved textbooks will then be established for the ensuing school year.

BERTRAM TRACHTENBERG
From the *Newark Mathematics Teacher*
December, 1952

THE ROUND ROBIN

MY STUDENTS and I were elated today because the long-awaited Round Robin just flew in. We had been looking forward to its arrival ever since we sent it off September 8. It has grown larger at each stop and now at last it is back home again.

This Round Robin took form last summer as I became interested in some projects that students from Frances Story's classes in St. Charles, Missouri, had made. We were both attending the last Mathematics Institute to be held at Duke University. Miss Story's students had made many illustrations of ideas and principles as they had learned about them. She had brought pictures of these with her. Since the supervisors of my state had been interested enough in my students' projects of a similar nature to invite me to one of

their meetings to give a demonstration of how I used them, I thought she might be interested in exchanging projects. I suggested that we start exchanging projects, or ideas for projects, as we and our classes thought of them.

I had long cherished the hope of making it possible for my students to see the best work of other students elsewhere in the United States. Mentioning this to Miss Story was the right thing to do. She was very enthusiastic about the idea. We both felt that the usual mathematics exhibits were enjoyed by teachers, but the pupils did not get a chance to see other pupils' work very often. Since Dr. W. W. Rankin, the leader of the Institute, inspired us to carry out our original ideas, we began planning right away to make this possible.

The difficulty of packing the projects and sending them seemed insurmountable, so we decided to send pictures of them instead. Frances Story suggested that we also send our best daily work. A few days after we had mapped out our plans we were talking about them to two other institutors, Margaret Striegel and Catherine Doyle. They immediately asked if they could exchange work with us. That is how the idea of the Round Robin came to our minds. We thought it would facilitate mailing if we made a Round Robin of it. Several others heard of our plan and asked if they might join. We told Veryl Schult, Supervisor of Mathematics in Washington, D.C., of our plans, and knowing how much zest she can add to an experiment, we asked her to join us. Our circle had grown to ten and we decided ten would be enough, since in going from one to another of us the Robin would travel by a circuitous route from Salisbury, Maryland, to Oregon, and back again, picking up material on the way. Specific plans were made by correspondence after the close of the Institute. The following excerpt from the letter which was sent to each Round Robiner explains the details of our plans.

"Dear Round Robiner:

. . . At the same time that I am mailing this letter to each of you, I am mailing the first package to Frances Story, who is the co-planner in this venture. So on September 8, 1952, the 'little Robin' takes flight. Please let's help it make the round safe and sound, growing bigger and bigger at each flight.

I would guess that the first thing you would like to know is who is in the round, and, second, to whom you will send the package. This information I am giving below:

Helen Warren to Frances Story
. . . September 8

Frances Story to Catherine Doyle
3528 North 57th Street
Milwaukee 16, Wisconsin

Catherine to Margaret Striegel
2247 North 73rd Street
Wauwatosa 13, Wisconsin

Margaret to Lesta Hoel
2138 N. E. 18th Street
Portland 12, Oregon

Lesta to Lurnice Begnaud
Box 557
Lafayette, Louisiana

Lurnice to Lucille Gilstrap
Senior High School
Dalton, Georgia

Lucille to Katherine Michel
Roanoke, Alabama

Katherine to Hope Rollins
209 West Wilson Street
Farmville, North Carolina

Hope to Veryl Schult
Wardman Park Hotel
Washington 8, D.C.

Veryl to Helen Warren
Snow Hill, Maryland

Don't you think five days is long enough for anyone to keep it? Allowing five days for each person and time for transportation, you can estimate just about when Robin will get to you. . . .

If you have other suggestions as to how this round could be improved, please send them to Frances or to me.

If you have time, I think it would add to our fun if when you receive the package you mail a card to each one of us. In that way we can all keep up with the whereabouts of Robin. You might also include a very brief summary of how you used the contents.

It is with a great deal of anticipation that I send 'Baby Robin' off on its maiden flight. Many good wishes for a Happy Landing.

Sincerely,
Helen Warren
in co-ordinate with Frances Story"

It is hard for me to realize now the misgivings I had when I packed the first materials to travel around this great big country of ours. It is not as difficult to recall how anxiously I waited for the first news of it from Miss Story. The enthusiasm was contagious and ran like wildfire through my students when we got the letter reporting the first safe landing of Robin. Thus the enthusiasm grew as we followed the Robin to Wisconsin where the Misses Doyle and Striegel took it to a meeting of mathematics teachers, after that to Oregon, and on through the various states.

All the while we kept wondering what new ideas we would get from the other schools. We were becoming more serious about our second contribution, hoping it would be as helpful and inspiring to others. Imagine our delight when we got a long letter from Veryl Schult, the last one on the list, telling us how she had taken Robin with her to the Christmas meeting of the National Council of Teachers of Mathematics in Lincoln, Nebraska. Then, right in the wake of the letter came Robin himself. The pupils couldn't wait to open it. It was really fun unpacking the long-awaited Robin. We eagerly examined its contents.

Every Round Robiner had contributed handsomely of the things that they thought would be most helpful. Knowing that other teachers and students would wish to see them, we arranged a very attractive bulletin board and table display. Miss Michel in Alabama had also arranged the contents of Robin as a display, open to the public. Students acted as attendants and explained the exhibit to visitors. Although I could not mention all of them, I am listing many of the materials in the following outline:

I. Illustrative material made by the students.
  A. Cartoons on:
    1. keeping an equation balanced.
    2. treating the expression enclosed in parentheses as one term. (My students have referred to this frequently.)
    3. collecting similar terms.
    4. directed numbers.
  B. Original poems.
  C. Puzzles:
    1. crossword.
    2. progressive blocks.
    3. algebraic expressions which, when simplified, form words in a sentence.
  D. Time exposures taken of a moving speck of light (one of a flash of light and one of a star) illustrating locus.
  E. Many studies in geometric form.
  F. Theorems illustrated with interesting cutouts.
  G. Cards decorated with geometric designs, stenciled.
  H. Slide rule.
  I. Vernier scale.
  J. Device showing that the side opposite the 30° angle of a right triangle equals ½ the hypotenuse.
  K. Curve stitchings.
  L. Transparencies for windows.
  M. Magic square design hand-tooled on a billfold with directions.
II. Teaching devices—teacher or commercially made.
  A. Curve stitching unit from "Things of Science."
  B. Signal Corps Posters.
  C. The Perspectograph—used to facilitate the drawings for solid geometry. You can obtain these from Miss Margaret Joseph, 1504 N. Prospect Avenue, Milwaukee, Wisconsin, at 65¢ each.
  D. Crystal of Calcite.
  E. Box of solid geometry plates with a pair of radio-flex glasses to view them (obtained from Newsom and Company, New York).
  F. Device illustrating color and balance.
III. Best daily work.
  A. Parabolas plotted.
  B. Points plotted which, when connected in order, make a picture.
  C. Magic square designs.
  D. Compilation of student's statements on "What I learned from curve stitching."
  E. Studies in inductive reasoning.
  F. Constructing one line parallel to another, each step on a different transparent sheet.
  G. Army insignia constructed.
IV. Pictures.
  A. Christmas tree which stood in the lobby of the hotel where the National Council of Teachers of Mathematics met in Lincoln, Nebraska, trimmed with polyhedrons and other three-dimensional figures.
  B. Student-made transit.
  C. Wind tunnel.
  D. Flying saucers.
  E. Conic sections.
  F. Cross section of student-made auto-

(*Continued on page 520*)

APPLICATIONS

FRANK B. ALLEN, *Guest Editor*
La Grange, Illinois

Al. 20 Gr. 9. *Functional Thinking*

The meat-packing industry obtains most of its beef cattle from the operators of feeding farms who make a business of fattening cattle for the market by means of scientifically controlled feeding programs. The operator of one such farm is trying to decide whether he should deal in premium grade beef or beef of lower quality which we will call "standard grade." The comparative facts are as follows: Standard grade beef cattle averaging 500 pounds per head can be purchased at fifteen cents a pound. After the feeding program is completed, they can be sold for twenty cents a pound, at an average weight of 1000 pounds per head. Premium grade cattle also average 500 pounds when purchased and 1000 pounds when sold, but in this case the initial cost is twenty cents per pound and the selling price is twenty-five cents. The cost of the feeding program is found to be substantially the same and the feeding capacity of the farm will handle as many of one kind as the other. Under these circumstances, which grade of beef will bring the greater profit? I have asked a good many people to consider this problem since it was first brought to my attention by Ralph Johnson, a member of our mathematics staff here at La Grange. The results have been most interesting. The majority, after a casual examination of the facts, conclude that it makes no difference which grade of beef is processed because the profit will be the same in either case. After all, the weight increments are the same, the differences between cost and selling price per pound are the same, and the costs of operation are supposed to be the same. Therefore the profits must be the same. This line of argument is so plausible and convincing that some people see no reason to bother with the simple arithmetic necessary for verification.

Those who do the arithmetic are sometimes surprised to learn that the profit on premium grade cattle is twenty-five dollars per head more than the profit on standard grade. Some who believed that the profits were the same are not convinced even after they are confronted with computation in which they can find no flaw. They still demand an explanation and this affords an excellent opportunity to focus attention on the principle applied here which turns out to be a very interesting example of functional thinking.

Let p represent the profit per head for premium grade expressed in dollars, and s the profit per head for standard grade. Then

$$p = 1000 \times \$.25 - 500 \times \$.20 = \$150$$
$$s = 1000 \times \$.20 - 500 \times \$.15 = \$125.$$

Suppose we let $a = 1000$, $b = 500$ and $d = \$.05$. The general formula for computing the profit then becomes: $a(x+d) - bx$ where x is \$.20 in one case and \$.15 in the other. If a, b, and d remain constant as they do in this problem, the value of this

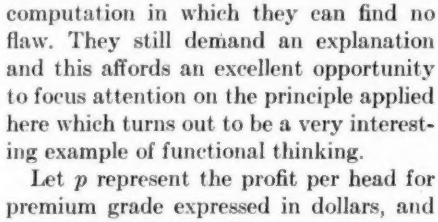

FIG. 1

expression depends upon x which represents the initial cost per pound. This function of x can be written: $(a-b)x+ad$. The value of this function evidently increases linearly with x so long as $a>b$.

For those who are visual-minded the situation can be pictured by a diagram.

The shaded area (Fig. 1) represents the profit and it is clear that this area will increase with x when a, b, and d remain constant.

S.G. 3 Gr. 10–12. *Computing the Size of a Room*

We know that when the sum of two numbers is fixed, the sum of their squares will decrease as the numbers approach equality, and will be a minimum when they are equal. The following problem provides an interesting application of this principle and involves some rather challenging puzzle elements as well.

A room is 30 feet long, 20 feet wide, and 10 feet from floor to ceiling. Find the length of the shortest line which can be drawn on the interior surface of the room (floor, walls, ceiling) from one corner of the ceiling to the diagonally opposite corner of the floor.

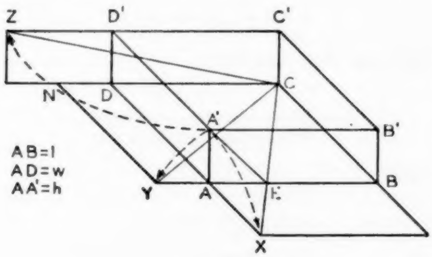

FIG. 2

Most students will compute the length of a line which follows the intersection of two walls and a diagonal of the floor. They are usually quite surprised to learn that this is by no means the shortest distance. Class discussion will eventually develop the idea that one wall of the room can be folded as shown in the diagram so as to become co-planar with the floor or some other wall. We cannot be sure that we have the correct result until all three essentially different methods of folding have been examined. When we study the three right triangles CNY, CDX, and $CC'Z$, we see that the sum of the legs is the same for each. We verify by computation that the sum of the squares of the legs is least for the triangle CDX where the legs are equal. Therefore the shortest distance in this case is $CE+EX=CE+EA'=42.4$ feet. In this way the student is led inductively to surmise the truth of the principle stated above.

It is interesting to generalize this problem by considering a room whose dimensions are l, w and h with $l>w>h$. The sum of the legs of all three triangles will be $l+w+h$. It can be easily verified that d the minimum distance required is given by the formula:

$$d=\sqrt{(h+w)^2+l^2}.$$

This distance corresponds to the line CX in our diagram. It is less than the length of CY for example because

$$\sqrt{(h+w)^2+l^2}<\sqrt{(h+l)^2+w^2}$$

or $h^2+2hw\quad w^2+l^2<h^2+2hl+l^2+w^2$ because by hypothesis $w<l$.

This problem can be stated in a form more appropriate for a Solid Geometry class as follows: How can we pass a plane through two diagonally opposite vertices of a rectangular solid whose dimensions are l, w, and h, $(l>w>h)$ so that the perimeter of the cross section shall be a minimum?

T. 7. Gr. 11–12. *Application of Parallelogram Law*

The following example may be of interest to teachers who are seeking variety in problems whose solutions require an application of the parallelogram law for the resolution of forces.

A weight of 100 pounds is supported as shown in Figure 3 in which AB is horizontal, $\alpha=40°$ and $\beta=20°$. Find the tensions t_1 and t_2 in the ropes AC and BC.

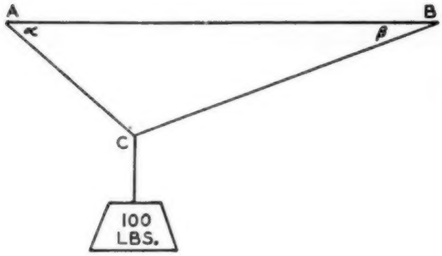

FIG. 3

Since there are several ways to solve this problem, a method can be chosen which is appropriate to the grade level of the class.

One method requires only the use of a simple scale drawing. The equilibrant must be represented by a line directed vertically upward whose length represents 100 on the scale we choose. Let this line be OW in our diagram (Fig. 4). We must now construct a parallelogram with diag-

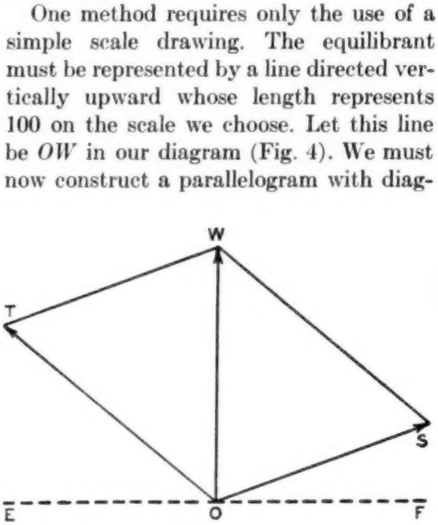

FIG. 4

onal OW and with sides OS and OT making angles of 20° and 40° respectively with the horizontal line EOF. The angles which these sides make with the vertical line OW are evidently 70° and 50° respectively. So we construct the triangle TOW having two given angles and the included side. We measure the lengths of OT and TW and apply our scale to find the required tensions t_1 and t_2.

Those who have had the necessary instruction in trigonometry will want to solve the problem by means of the sine law. The problem can be made more interesting for a trigonometry class by giving the lengths of the ropes and the distance between supports so that the angles α and β must be determined before the sine law can be applied

It is also instructive to develop a general formula for t_1 and t_2 in terms of α, β and the weight w. In Figure 4 we now have $OW=w$, $<TOW=90-\alpha$ $<TWO=90-\beta$ and $<OTW=\alpha+\beta$

Applying the sine law to triangle TWO we have

$$OT=t_1=\frac{w\cos\beta}{\sin(\alpha+\beta)} \quad \text{and}$$

$$TW=OS=t_2=\frac{w\cos\alpha}{\sin(\alpha+\beta)}$$

We may also solve the problem by an application of the theorem: "A necessary and sufficient condition that the resultant of n concurrent forces be zero is that the sums of their components along each of two perpendicular lines equal zero." Using EF and OW as our perpendicular lines, we obtain two equations in two unknowns.

$$t_1\cos\alpha=t_2\cos\beta$$

$$t_1\sin\alpha+t_2\sin\beta=w$$

Solving these we have:

$$t_1=\frac{w\cos\beta}{\sin\alpha\cos\beta+\cos\alpha\sin\beta}$$

and

$$t_2=\frac{w\cos\alpha}{\sin\alpha\cos\beta+\cos\alpha\sin\beta}.$$

These results are identically equivalent to those obtained by use of the sine law.

Additional problems showing the application of elementary vector analysis to situations involving tensions, friction, torque, etc. can be found in Chapter six of "Plane Trigonometry," by Corliss and Berglund (Houghton Mifflin, 1950), from which the above theorem is quoted.

REFERENCES FOR MATHEMATICS TEACHERS

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Just What Is Mathematics?

To state adequately and in simple terms the meaning of some very fundamental notion such as that of number, or space, or mathematics, is, of course, virtually impossible. Yet the layman and the novice are entitled to a reasonable answer to the question: just what *is* mathematics? And, as teachers, we should be able to give such an answer.

As a matter of fact, I often give my students in the teaching of mathematics precisely this task as their first assignment. I ask them to write a short paper (not more than 1250 words) in simple, informal language, setting forth the gist of mathematics, its intrinsic nature, its quality, its very essence—just what is this thing called mathematics? In giving this assignment, they are urged *not* to list and define the various branches of mathematics. They are cautioned, too, against trying to give formal definitions of mathematics. They are warned that it is definitely a difficult assignment.

Although these students are mathematics majors who have already had from 12 to 24 hours in advanced electives beyond elementary calculus, I rarely get a really satisfying paper. To be sure, the entire set of papers invariably furnishes an excellent basis for subsequent class discussion. Perhaps some of my readers would like to try to characterize mathematics, to describe its spirit, and to identify the hallmarks of the Queen of the Sciences before turning to the literature.

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(*Continued on page 521*)

AIDS TO TEACHING

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BOOKLETS

B. 174—Developing Meaningful Practices in Arithmetic

Central New York School Study Council, 201 Slocum Hall, College Place, Syracuse 10, New York.

Booklet; $5\frac{1}{2}'' \times 8\frac{1}{2}''$; 123 pages; $2.00.

Description: This booklet is one of the results of the Central New York School Study Council and was produced through cooperation of many members of school systems. These teachers working together turned out some very practical material for arithmetic. The introductory chapters tell of their plan of working together to improve instruction and tell how the present report was put together. Then three chapters on "Meaningful Practices" take up the early elementary, middle elementary, and the upper elementary individually. Finally, the results of working together are summarized.

Appraisal: The major part of the book is in these three chapters on "Meaningful Practices." Each of these chapters begins with a section telling the basic mathematical understandings which the group felt necessary. Then the understandings are taken up separately giving teaching techniques and text items which should contribute to those understandings. The most recent philosophy and methods of teaching are incorporated, and the extreme practicality and classroom usefulness of this material would appeal to many teachers. Certainly this booklet will obtain its greatest usefulness by being put into discussion groups similar to that in which it was formed. By having arithmetic teachers analyze, discuss, and apply the message described here, arithmetic teaching should be influenced for the better throughout the whole country.

EQUIPMENT

E. 130—Place Value Board

Ideal School Supply Company, 8322 Birkhoff Avenue, Chicago, Illinois.

Model; wood base; 10 wire loops; 10 beads on each loop; comma insert; decimal insert; $3.75

Description: The purpose of this manipulative device is to visualize place value in our number system. It is similar to an abacus in that discs are manipulated on a wire to picture the meaning of a number from zero through billions. It is different from an abacus in that the vertical wires are looped 180° at the middle with a panel between the two sections of wire. In this way only the discs on one side of the loop are visible. It has ten wires with ten discs on each wire permitting the reading of ten place numbers. A decimal insert can be used to help represent decimal numbers. Comma inserts facilitate the reading of large numbers.

Appraisal: This place value board has two distinct advantages over an ordinary abacus: It permits the wires to be vertical so that the place value of the number represented corresponds to place value in written numbers, and the unused discs are out of sight and thus cannot distract

the attention from the discs in use. Another advantage of this arrangement is that this board can be used in an abacus for any number system with a base less than eleven.

E. 131—The Arithmetic Educator

Huber Pipe Organ Manufacturing Company, Trempealeau, Wisconsin.

Device; electric symbol board; 57″×18″; keyboard; $224.00

Description: This device consists of a symbol board which is to be mounted on the wall of the classroom. The board has two horizontal rows of numbers from 0 through 12 with the symbols +, −, × between the rows. Electric lights are mounted under each number and symbol. A small keyboard on the teacher's desk is used to control the lights behind the symbols. Thus it is possible to light numbers and symbols for all number combinations for addition, subtraction, and multiplication from 0 through 12. The device is guaranteed for ten years.

Appraisal: Although an electric device such as this would be convenient and of high interest value in the classroom, few schools will have sufficient funds to purchase one as a substitute for drill cards.

E. 132—Hundred Spool Number Board

E. 133—Primary Spool Number Board

The John C. Winston Company, 1010 Arch Street, Philadelphia 7, Pennsylvania.

Number board; 16″×16″; 100 spools $\frac{3}{4}$″ in diameter; $4.75 each

Description: The number boards have 100 pegs arranged in ten rows of ten pegs in a row. The spools are wood cylinders which fit over the pegs. A card accompanying the hundred-spool board gives the place value of each column by placing it above the ten columns of spools. This facilitates the reading of large numbers. The only difference between the hundred-spool board and the primary spool board is that the primary spools are colored and the board does not have the place-value card. The number board can be used by placing it on a chalk tray or laying it flat on a table.

Appraisal: By placing spools on pegs, a variety of patterns can be arranged to show the meaning of numbers, number combinations, place value, and locations. The board can be used to discuss the relationships of addition to subtraction, multiplication to subtraction, and decimals to per cent. Since the board is small for class demonstrations, it will be most useful for small groups or individual pupils. Although a teacher could readily make a board of this type with plywood, nails, and empty thread spools, it is hardly necessary when a well-finished board can be purchased at this price.

FILMS

F. 95—Meaning of Number Series: What Are Decimals

Films, Inc., 1150 Wilmette, Wilmette, Illinois.

Film; black and white; one reel, 10 minutes; $45.00 ($2.50, 1–3 days rental)

Description: This film gives a review of common fractions, and tells about the uses of tenths, hundredths, and thousandths. The decimal point is introduced, and the reading and writing of decimals. Mixed decimals are followed by a short discussion of the importance of placing the decimal point correctly. Finally the relationship between the common and decimal fraction is explained.

Appraisal: There are certainly many excellent examples of decimals in this film and very good, adequate material to introduce the subject. Sometimes the animation is poor and distracting rather than helpful. Occasionally, there is a mechanical explanation of reading decimals, for example, .125, rather than a rational explanation which should be used at an introductory level. However, this is a good film and will have many uses in the teaching of arithmetic.

F. 96—Meaning of Number Series: What Are Fractions

Films, Inc., 1150 Wilmette, Wilmette, Illinois.

Film; black and white; one reel, 10 minutes; $45.00 ($2.50, 1–3 days rental)

Description: Fractions are treated by explaining to us the meaning which they have as "part." We are then told that the size depends upon the denominator, and the meaning of numerator and denominator is explained. Both the "part of the whole" and the "part of the group" ideas are introduced.

Appraisal: The slow speed at which ideas are introduced in this film is certainly to its advantage. Some attempt is made to get participation from the audience. On the other hand, much of the type of work explained could be done on the blackboard, some of the labels in illustrations can be confused with lengths of lines, and there is a certain monotony in the treatment because one is always cutting wood to explain the meaning of the fraction. This is a good teaching film but must be supplemented by much careful classroom work.

F. 97—Using the Bank

Encyclopædia Britannica Films, Inc., 1125 Central Avenue, Wilmette, Illinois.

Film; 16 millimeter, sound; one reel, 10 minutes; black and white; $50.00 ($2.50, 1–3 days rental)

Description: In this film we hear about many topics such as savings accounts, the way the bank keeps records, how banks invest their money, how one may borrow money from banks, and the process of using checking accounts. Several mathematical ideas are involved: the bar graph of deposits, the idea of interest, a "Proving Machine" for checking the various deposit slips, how interest is calculated by tables, and discount; also how to make out checks and stubs, how the bank processes checks, and the federal bank system.

Appraisal: Very useful material is covered in this film and the language is suitable for the grade level intended. The acting is adequate and there is just enough plot to tie the subject together. One will sense that by trying to cover a great many functions of the bank the treatment is very full. But, by proper preparation of the class, this should certainly not be a disadvantage. This film is to be recommended.

FILMSTRIPS

*FS. 171 through FS. 176—Light on Mathematics—Arithmetic**

FS. 171—Five Keys to Mathematics (48 frames)
FS. 172—Addition and Subtraction (33 frames)
FS. 173—Multiplication and Division (75 frames)
FS. 174—Addition and Subtraction of Fractions (47 frames)
FS. 175—Square Root and Cube Root (52 frames)
FS. 176—Multiplication and Division of Fractions (30 frames)

Jam Handy Company, 2900 E. Grand Boulevard, Detroit 11, Michigan.

B & W ($4.00 each).

Description of FS. 171: Our world is highly dependent on the work of specialists. Specialization makes it possible to perform tasks accurately, efficiently, and economically. Precision in thought and action should result from the study of mathematics. To be a success on the job or in the home we need to remember the principles we have learned and to use ingenuity in applying these principles to new situations. The five keys to learning mathematical principles are the following: (1) Use your hands to make problems concrete by making drawings or models, (2) Prove the principle several ways, (3)

* See also *FS. 108*, Fractions, Decimals and Percentage and *FS. 109*, Order of Operations, May 1952.

State the principle in your own words, (4) Make a list of job applications, (5) Use a reference book.

Description of FS. 172: This filmstrip reviews the fundamental processes of arithmetic. It emphasizes the meaning of numbers so that pupils will understand the mechanics of carrying and borrowing. Checking computations can be done by operating in the reverse order or by estimating the answer. The illustrations and operations are verbal and symbolic with little or no attempt being made to illustrate the meaning of numbers or operations with concrete materials.

Description of FS. 173: The emphasis of this filmstrip is on meaning and understanding of numbers and processes, as it shows how to multiply and divide. The meaning of partial products and quotient figures are discussed as the process is shown. Methods of checking include the casting out of nines. Some short cuts in multiplication are illustrated.

Description of FS. 174: The method of adding quantities measured in the same units is used as an introduction to the addition of fractions having the same denominator. The denominator is shown as determining the size of the fraction. The sizes of different fractions are illustrated by sections of equal circles. In a method similar to arithmetic textbooks the strip discusses finding the lowest common denominator by repeated divisions, the addition of mixed numbers, and borrowing in subtraction of fractions. The anticipation of the result is shown as a method of checking the answer.

Description of FS. 175: If a number is represented by a length, then multiplication can be represented by a rectangle and the square of a number by a square. This forms the basis for visualizing the meaning of square root and of each step in extracting the square root of a six-digit number. Similarly a cube is divided into cubes and rectangular prisms to illustrate each step in computing the cube root of a number. Most high-school students will probably get lost in these three-dimensional representations.

Description of FS. 176: This filmstrip begins by reviewing the meaning of a fraction and illustrating the meaning of multiplication as related to finding the area of a rectangle. Examples are then given which show how to use the rules for the multiplication and division of fractions. The inversion of the divisor is explained on the basis of division being the opposite of multiplication. The division of a whole number by a fraction is illustrated by comparing a train and tunnel whose lengths are equal. The train "going into" the tunnel illustrates $1 \div 1 = 1$!

Appraisal of FS. 171 through FS. 176: Terms such as "cancellation" and "goes into" which are no longer considered appropriate are used frequently in these films. Rules and results of computations are often stated without any attempt to show how these rules or results might be discovered by the viewer. Thus it would seem that these films are out of tune with accepted principles of learning. Most of the content is similar to textbook treatment with much verbal material included.

What Is Going on in Your School?

(*Continued from page 511*)

mobile cylinder with movable piston illustrating the sine curve.

G. Device for showing the line value of functions of an angle.

V. Teaching units.
  A. Unit on complex numbers.
  B. Unit on crystallography.
  C. Unit on taking math from the reading of the newspaper.
  D. Mathematics in aviation.

Among those who came to see our display, we are proud to list our Principal, Mr. Jones; County Supervisor, Miss Helen Wootton; and my State Supervisor, Mr. Willis White.

Robin has grown to a fledgling now and is taking its second trip in a wooden box with space for three-dimensional projects.

If this second round is a success, we plan to get a mailing box to facilitate the mailing of these projects. With this second round, we have included a Mathematics newspaper which the students of Wi-Hi have named "The Stag," the name made from the first letter of Students, Trigonometry, Algebra, Geometry, and General Mathematics. We hope to have it grow with the Robin. We are also sending two victrola records on which the students recorded their voices discussing "The Importance of Mathematics." This, when added to by the other Round Robiner's students, will, we think, become more and more interesting.

Needless to say, this experiment has been invaluable in many ways. All of us agree that it is a strong incentive to do better work. For many years I have been trying to make my students realize that they are only one part of a big organization of classes over the United States, and that Mathematics is a universal subject enjoyed by students all over the world. Round Robin is helping to instill this "One World" idea in them.

Miss Begnaud sums up its value by saying, "The Robin serves as an inspiration to the students. It makes them want to improve so as to do as well as others whose work they've seen."

That is the story that explains why we were so elated when Round Robin flew in.

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(*Continued from page 516*)

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(*Continued on page 524*)

BOOK SECTION

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BOOKS RECEIVED

Elementary

Arithmetic for Today, Workbooks 1 through 8, Thomas J. Durell, Adaline P. Hagaman, James H. Smith. Columbus, Charles E. Merrill Books, 1951. Paper, all books have 64 pages, Grade 1, 48¢, Grade 2, 52¢, Grades 3–6, 56¢, Grades 7–8, 56¢.

Arithmetic for Today, (Clothbound editions for Grades 3, 4, 5, and 6), Thomas J. Durell, Adaline P. Hagaman, James H. Smith. Columbus, Charles E. Merrill Books, 1953. Cloth, all books have iii+216 pages, $1.32.

Teaching Children Arithmetic, Primary, Intermediate, Upper Grades, Robert Lee Morton. New York, Silver Burdett Company, 1953. Cloth, v+566 pp., $4.50.

Secondary

Functional Mathematics, Books 1 and 2, William A. Gager, Charlotte Carlton, Carl N. Shuster, Franklin W. Kokomoor. New York, Charles Scribner's Sons, 1953. Cloth, iii+447 pp., $2.96.

College

A First Course in Functions of a Complex Variable, Wilfred Kaplan. Cambridge 42, Massachusetts, Addison-Wesley Publishing Company, Inc., 1953. Cloth, v+619 pp., $3.50.

A Survey of Modern Algebra, Garrett Birkhoff and Saunders MacLane. New York, The Macmillan Company, 1953. Cloth, v+469 pp., $6.50.

Trigonometry, John F. Randolph. New York, The Macmillan Company, 1953. Cloth, v+220 pp., $3.00.

Miscellaneous

Youth—The Nation's Richest Resources Their Education and Employment Needs, The Interdepartmental Committee on Children and Youth of the Federal Government. Washington, D. C., United States Government Printing Office, 1951. Paper, iii+54 pp., $.20.

REVIEWS

The Algebra of Vectors and Matrices, T. L. Wade. Cambridge, Massachusetts, Addison-Wesley Press, Inc., 1951. ix+189 pp., $4.50.

The material in this textbook is accessible to students familiar with determinants and analytic geometry. It is written on an elementary level, and presents motivation frequently, usually by discussing vectors in physical space. Definitions are often given for the two and three dimensional cases before being generalized, and the most general space considered at any length is the set of ordered n-tuples with entries from a field. In the discussion of matrices, the entries are usually assumed to be real numbers. The author is consistent in his interpretation of vectors as position vectors and of linear transformations as mappings of points, merely mentioning the other aspects of these concepts.

The attempt to retain an elementary tone leads to certain faults and other situations regarded by the reviewer as undesirable: technical terms are occasionally used before they are defined; some of the definitions are incomplete or inaccurate; it is sometimes difficult to tell when the discussion changes over to n-dimensional space from the two or three dimensional example; the definitions of matrix, sum of matrices, and product of matrices are all motivated by the concept of linear transformation, yet the corresponding definitions for linear transformations are not made until much later. On the other hand, an elementary exposition of this nature fills a definite need, and might possibly be used to introduce the student to concepts of modern algebra soon after the freshman course in analytic geometry.

The contents include among other things discussions of vectors, matrices, groups, transformations, systems of linear equations, linear, bilinear, and quadratic forms, cogredience and contragredience, and a short chapter on applications of matrix algebra.—T. C. HOLYOKE, Northwestern University, Evanston, Illinois.

Applied Mathematics for Technical Students (Revised ed.), Murlan S. Corrington. New York, Harper & Brothers, 1952. Cloth. xiii +277 pp. text+135 pp. tables, $4.00.

The first edition of *Applied Mathematics for Technical Students* was published in 1943. This revised edition contains a new chapter on portions of geometry essential to the solutions of practical problems. It contains also a new chapter on graphic methods in which log, semilog, and polar graph paper are discussed. Other new features include simplified methods of extracting

square and cube roots, and the extension of trigonometry so that right triangles can be solved without tables.

This book is written primarily for trade schools, factory training courses, or for pre-engineering students. The mathematical theory is presented in a clear-cut manner. The many ways in which the author has projected the theory into the solution of practical problems as found in modern industry is to be commended.

While the book is well organized, well written, and free from disturbing errors, on page 17 it does not present the best practice where it suggests adding zeros to fill out numbers, and at the top of page 24 where measurements have been recorded to 0.01 inch and their average has been given to 0.001 of an inch. On page 39 and throughout the book, if the author had a purpose in omitting the units of measurement from the numbers placed on the drawing, he should have stated his purpose. Other items such as these could be mentioned but they are secondary to the fact that this is a good book for all secondary mathematics teachers to have available.—William A. Gager, University of Florida, Gainesville, Florida.

Basic Mathematics for Engineering and Science, Walter R. Van Voorhis and Elmer E. Haskins. New York, Prentice-Hall, Inc., 1952. x+567 pp., $5.75.

This text is an attempt to coordinate in a unified analytic treatment the essential topics in college algebra, trigonometry, and analytic geometry which would meet the needs of students who are majoring in engineering, physics, chemistry, and other sciences.

The subject matter treats the usual topics of first-year college mathematics: functions and functional relationships, linear functions and systems of linear equations, quadratic and polynomial functions, exponential and logarithmic functions, complex numbers, conic sections, locus problems, series, probability and statistics.

One of the difficulties found in most so-called "unified" texts is that the student fails to find anything "unifying" about the text he is using. In this text, however, the function concept is used to do a reasonably good job of unifying the topics which constitute elementary college mathematics.

In the chapter on coordinate geometry the author follows a modern trend of treating polar coordinates along with rectangular coordinates. The frequent use of the summation symbol $\sum$ should prepare the student for the introduction of this symbol when he encounters it in the calculus. A brief treatment of the operation and construction of a slide rule is included in the chapter on logarithms. In the chapter on the general equation of the second degree, the authors should have employed a matrix transformation for handling rotation of axes.

The appendix contains tables of powers and roots, common logarithms, natural logarithms, trigonometric functions, logarithm of the trigonometric functions, radians to degrees and degrees to radians, values of e^x and e^{-x}.

The choice of problem material is excellent. Applications from engineering and science are used in abundance. The length of the book makes it possible to include enough varied materials for instructors to select subject matter to meet the needs of the various classes of students. This book should be well received by students and instructors alike.—Irwin K. Feinstein, Chicago Undergraduate Division, University of Illinois.

Calculus and Analytic Geometry, George B. Thomas, Jr. Cambridge, Massachusetts, Addison-Wesley Press, Inc., 1951. 685 pp., $6.00.

The author in this book presents the material in a very readable manner. The drawings are an excellent feature. There are also numerous illustrations of the subject matter with sufficient problems for drill.

I like many of the features on integration, especially the use of inverse hyperbolic functions. It is to be regretted that the easy rule for obtaining the values of constants in partial fractions with distinct linear denominators is omitted.

An excellent feature of the book is the development of the analytic geometry as needed for the calculus. Teachers usually must make this development whatever the text he may use.

Unless this book is to be used for work beyond the elementary courses, such topics as vector analysis and complex variable might readily be omitted.—Herbert L. Lee, University of Tennessee, Knoxville, Tennessee.

Calculus and Analytical Geometry, Cecil Thomas Holmes. New York, McGraw-Hill Book Co., 1950. x+416 pp., $4.75.

This book fulfills a need in the undergraduate curriculum for non-mathematics majors. The text very aptly combines two fundamental courses without any serious loss of completeness with the stress placed on the calculus. It would seem that the book is particularly suited to engineers who need an early introduction to the calculus in their course work.

The book is clearly written, adequately illustrated, and has a liberal number of problems with answers in the back. There is very little loss in mathematical rigor and the topics are so arranged that one, two, or even three terminal mathematics courses can be taught using the text as a guide.—Emil J. Walcek, Parks College of Aeronautical Technology, East St. Louis, Illinois.

Elements of the Theory of Functions, Konrad Knopp. New York, Dover Publications, Inc., 1953. Paper or cloth. 140 pp. Paper $1.25; Cloth $2.25.

This is, to some extent, an introduction to the same author's *Theory of Functions* printed

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Mathematical Miscellanea

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This property is applied in the construction below.

In Figure 5 circle POQ represents a given circle whose center is lost; P and Q are its intersections with a circle whose center is O. PT is a diameter of (O). Cut (O) at R_1 using P as center, and at R_2 using Q as center, with QT as radius both times. Draw PR_1 and QR_2. Their intersection K is the center of the given circle.

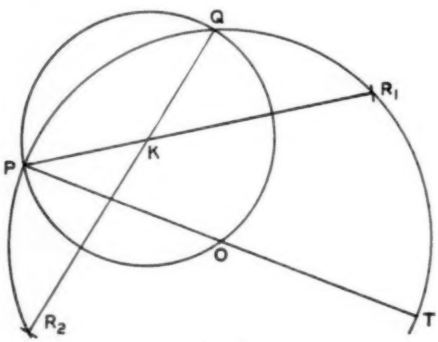
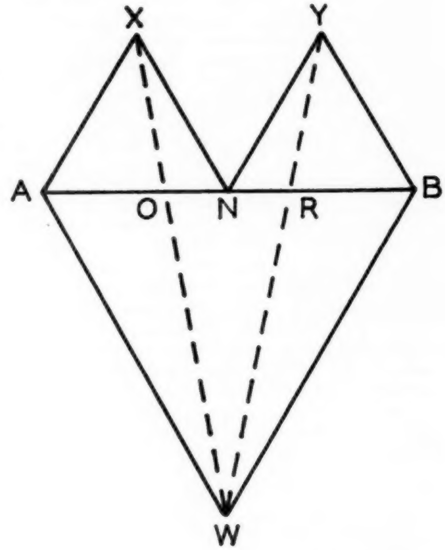


FIG. 5

89. A Novel Linear Trisection

The following construction was submitted by Robert J. Orr of Iowa State Teachers College. It was originated and proved by Miss Patty Hake, a student. We pass it on to the reader to prove and to generalize.



Let AB be a given line-segment. Find the midpoint N of AB. On one side of AB construct equilateral triangle ABW. On the opposite side of AB construct equilateral triangles ANX and NBY. Draw WX and WY, intersecting AB at O and R respectively, thus making $AO=OR=RB$.

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References for Math. Teachers

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Book Section

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by the same publishers. The purpose of the book is to give the reader a feeling for the new features that arise when functions over the complex numbers, rather than real functions, are studied. Particularly careful attention is given to the new geometric ideas involved. The fundamental notions of the following topics, among others, are discussed: topology of the complex plane, derivatives, power series, conformal mapping, rational and trigonometric functions, the concept of a Riemann surface.

It is assumed that the reader is familiar with the brand of real variable theory known in this country as advanced calculus. The book is written in a beautiful style for which appropriate credit is due to the translator; it is a pleasure to read, not only for the young student but also for the blasé who knows "all about" what is in it.—W. E. JENNER, Northwestern University, Evanston, Illinois.